Mathematical Economics
Lecture 1

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Syllabus

Mathematical Theory of Demand

Utility Maximization Problem

Expenditure Minimization Problem

Mathematical Theory of Production

Profit Maximization Problem

Cost Minimization Problem

General Equilibrium Theory

Growth Models

Dynamic Optimization

Syllabus

Mathematical Theory of Demand

- Budget Constraint
- Consumer Preferences
- Utility Function
- Utility Maximization Problem
- Optimal Choice
- Properties of Demand Function
- Indirect Utility Function and its Properties
- Roy’s Identity

Syllabus

Mathematical Theory of Demand

- Expenditure Minimization Problem
- Expenditure Function and its Properties
- Shephard's Lemma
- Properties of Hicksian Demand Function
- The Compensated Law of Demand
- Relationship between Utility Maximization and Expenditure Minimization Problem

Syllabus

Mathematical Theory of Production

- Production Functions and Their Properties
- Perfectly Competitive Firms
- Profit Function and Profit Maximization Problem
- Properties of Input Demand and Output Supply Functions

Syllabus

Mathematical Theory of Production

- Cost Minimization Problem
- Definition and Properties of Conditional Factor Demand and Cost Function
- Profit Maximization with Cost Function
- Long and Short Run Equilibrium
- Total Costs, Average Costs, Marginal Costs, Long-run Costs, Short-run Costs, Cost Curves, Long-run and Short-run Cost Curves

Syllabus

Mathematical Theory of Production

Monopoly

Oligopoly

- Cournot Equilibrium
- Quantity Leadership – Slackelberg Model

Syllabus

General Equilibrium Theory

- Exchange
- Market Equilibrium

Syllabus

Neoclassical Growth Model

- The Solow Growth Model
- Introduction to Dynamic Optimization
- The Ramsey-Cass-Koopmans Growth Model

Models of Endogenous Growth Theory

Convergence to the Balance Growth Path

Recommended Reading

- Chiang A.C., Wainwright K., *Fundamental Methods of Mathematical Economics*, McGraw-Hill/Irwin, Boston, Mass., (4th edition) 2005.

- Chiang A.C., *Elements of Dynamic Optimization*, Waveland Press, 1992.

- Romer D., *Advanced Macroeconomics*, McGraw-Hill, 1996.

- Varian H.R., *Intermediate Microeconomics, A Modern Approach*, W.W. Norton & Company, New York, London, 1996.

The Theory of Consumer Choice

- The Budget Constraint
- The Budget Line Changes (Increasing Income, Increasing Price)

- Consumer Preferences
- Assumptions about Preferences

- Indifference Curves: Normal Good, Substitutes, Complements, Bads, Neutrals
- The Marginal Rate of Substitution

Consumers choose the **best** bundle of goods they **can afford**

- How to describe what a consumer can afford?

- What does mean the best bundle?

- The consumer theory uses the concepts of a budget constraint and a preference map to analyse consumer choices.

The budget constraint – the two-good case

- It represents the combination of goods that consumer can purchase given current prices and income.
- (x_1, x_2), $x_i > 0$, $i = 1, 2$ - consumer's consumption bundle (the object of consumer choice)
- (p_1, p_2), $p_i > 0$, $i = 1, 2$ - market prices of the goods

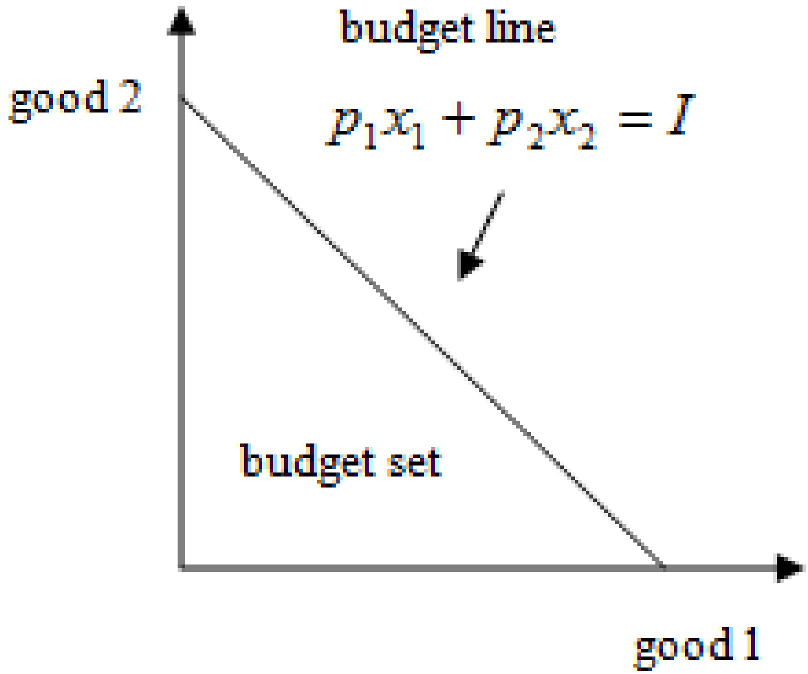
The budget constraint – the two-good case

- The budget constraint of the consumer (the amount of money spent on the two goods is no more than the total amount the consumer has to spend)

$$p_1x_1 + p_2x_2 \leq I$$

- $I > 0$ - consumer's income (the amount of money the consumer has to spend)
- p_1x_1 - the amount of money the consumer is spending on good 1
- p_2x_2 - the amount of money the consumer is spending on good 2

Graphical representation of the budget set and the budget line



- The set of affordable consumption bundles at given prices and income is called **the budget set** of the consumer.

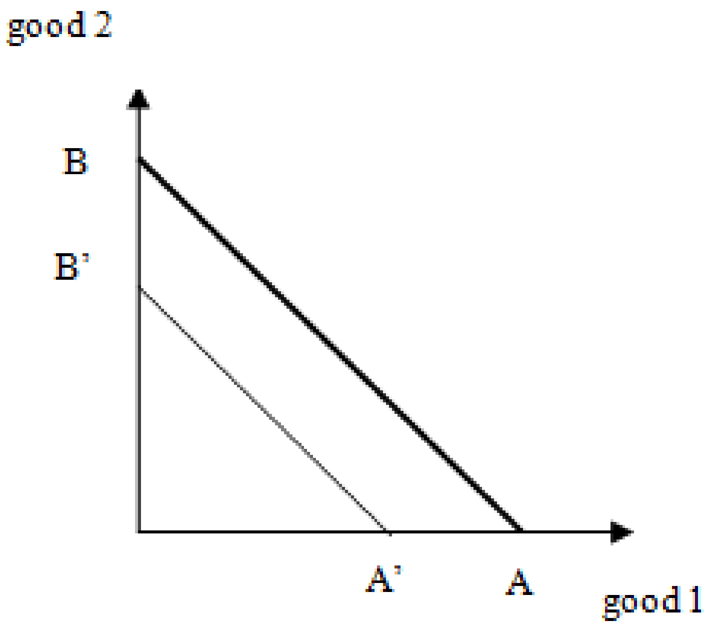
The Budget Line

The budget line has

- **a slope** of $-\frac{p_1}{p_2}$ - the opportunity cost of consuming good 1 (in order to consume more of good 1 consumer has to give up $\frac{p_1}{p_2}$ units of consumption of good 2) ($dx_2 = -\frac{p_1}{p_2} dx_1$)
- **a horizontal intercept** of $\frac{I}{p_1}$ and **a vertical intercept** of $\frac{I}{p_2}$ (measure how much consumer could get if they spent all income on good 1 and 2, respectively)

The Budget Line Changes

- Increasing (decreasing) income – an increase (decrease) in income causes a parallel shift outward (inward) of the budget line (a lump-sum tax; a value tax)



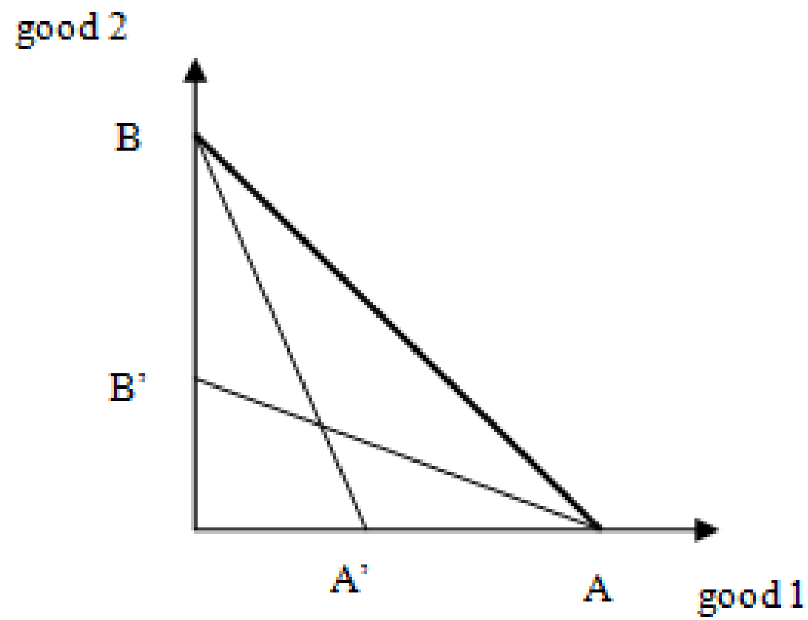
The Budget Line Changes

- Increasing price – if good 1 becomes more expensive, the budget line becomes steeper.
- Increasing the price of good 1 makes the budget line steeper; increasing the price of good 2 makes the budget line flatter.
- A quantity tax (excise)

  A value tax (*ad valorem* tax)

  A quantity subsidy

  Ad valorem subsidy



Exercise 1

The budget equation is given by $p_1x_1 + p_2x_2 = I$. The government decides to impose a lump-sum tax of T, a quantity tax on good 1 of t_1 and a quantity subsidy on good 2 of s_2. What is the formula for the new budget line?

Consumer Preferences

In order to describe the consumer's preferences over different consumption bundles in a systematic way we say that:

- $x \succ y$ - the bundle $x = (x_1, x_2)$ is strictly preferred to the bundle $y = (y_1, y_2)$ (consumer definitely wants the x-bundle rather than y-bundle; consumer always chooses x when y is available).
- $x \sim y$ - consumer is indifferent between two bundles; consumer would be just as satisfied consuming bundle x as they would be consuming y.
- $x \succsim y$ - the bundle x is weakly preferred to the bundle y (the consumer thinks that the bundle x is at least as good as the bundle y).

Consumer Preferences

$$P_s = \{(x, y) \in X \times X \ | x \succ y\} - \text{relation of strict preference}$$

$$I = \{(x, y) \in X \times X \ | x \sim y\} - \text{relation of } \textit{indifference}$$

$$P = \{(x, y) \in X \times X \ | x \succsim y\} - \text{relation of weak preference}$$

Assumptions about Preferences

- **Completeness:** for all x and y in X either $x \succsim y$ or $y \succsim x$ or both (any to bundles can be compared; the consumer is able to make a choice between two given bundles)
- **Reflexivity**: for all x in X, $x \succsim x$ (any bundle is as least as good as an identical bundle)
- **Transitivity**: for all x, y, and z in X, if $x \succsim y$ and $y \succsim z$ then $x \succsim z$ (the assumption is necessary for any discussion of preference maximization since if preferences were not transitive, there might be sets of bundles which had no best elements; the hypothesis about people's choice behaviour)

Assumptions about Preferences

- **Continuity**: for all y in X $\{x: \ x \succsim y\}$ and $\{x: \ x \precsim y\}$ are closed sets. It follows that $\{x: \ x \succ y\}$ and $\{x: \ x \prec y\}$ are open sets.

The assumption says that if (x^i) is a sequence of consumption bundles that are all at least as good as a bundle y and if sequence converges to some bundle x^* , then x^* is at least as good as y.

(If y is strictly preferred to z and if x is a bundle close enough to y, then x must be strictly preferred to z)

Assumptions about Preferences

- **Weak monotonicity**: if $x \geq y$ then $x \succsim y$ (at least as much of everything is as least as good)
- **Strong monotonicity:** if $x \geq y$ and $x \neq y$, then $x \succ y$ (at least as much of every good, and strictly more of some good, is strictly better (when a goods are good)). Strong monotonicity is not satisfied for garbage, pollution (i.e. for bads).
- **Weak convexity:** given *x*, *y*, and *z* in *X* such that $x \succsim y$ and $y \succsim z$, then it follows that $\lambda x + (1-\lambda) y \succsim z$ for all $\lambda \in (0,1)$.

Assumptions about Preferences

- **Strong convexity:** given $x \neq y$ and z in X, if $x \succsim y$ and $y \succsim z$ then $\lambda x + (1-\lambda) y \succ z$ for all $\lambda \in (0,1)$.

1. $\forall x, y \in X \quad x \succ y, \ \ x \neq y \ \Rightarrow \ \lambda x + (1-\lambda) y \succ y$;
2. $\forall x, y \in X \quad x \sim y, \ \ x \neq y \ \Rightarrow \ \lambda x + (1-\lambda) y \succ y$;
3. $\forall x, y \in X \quad x \sim y, \ \ x \neq y \ \Rightarrow \ \lambda x + (1-\lambda) y \succ x$.

The relations of strict preference, weak preference and indifference are not independent concepts!

$x \succ y$ and $x \sim y$

$$x \succsim y \Leftrightarrow (x \succ y \vee x \sim y)$$

Theorem: $\forall x, y, z \in X$

- $x \succ y \vee y \succ x \vee x \sim y$;
- $(x \succsim y \wedge y \succsim x) \Leftrightarrow x \sim y$;
- $(x \succsim y \wedge \neg(y \succsim x)) \Leftrightarrow x \succ y$;
- $(x \succsim y \wedge y \succ z) \Rightarrow x \succ z$;

$$x \succsim y$$ $$x \sim y \Leftrightarrow \left(x \succsim y \wedge y \succsim x\right) \text{ and}$$ $$x \succ y \Leftrightarrow \left(x \succsim y \wedge \neg\left(y \succsim x\right)\right)$$
Theorem: $\forall x, y, z \in X$ • $x \succ y \vee y \succsim x$; • $\left(x \succsim y \wedge y \succ z\right) \Rightarrow x \succ z$; • $x \succsim y \Leftrightarrow x \succ y \vee x \sim y$; • $x \succ y \vee y \succ x \vee x \sim y$;

Exercise 2

Let assume that $X = \{a, b, c, d\}$ - the consumption set, a, b, c, d - consumer's bundles. Check whether the following relation of weak preference

$$P = \{(a, a), (a, b), (a, d), (b, b), (b, c), (c, a), (c, c), (d, b), (d, c), (d, d)\}$$

is complete and transitive.

Exercise 3

For the consumption set $X = \{a, b, c, d\}$ relations of preferences are describe as follows:

	a	b	c	d
a	$a \sim a$	$a \succ b$	$a \sim c$	$a \succ d$
b		$b \sim b$		$b \succ d$
c	$c \sim a$	$c \succ b$	$c \sim c$	$c \succ d$
d				$d \sim d$

Check whether

a) the relation of indifference is reflexive and symmetric;

b) the relation of strict preference is transitive;

c) the relation of weak preference is complete and transitive.

Indifference Curves

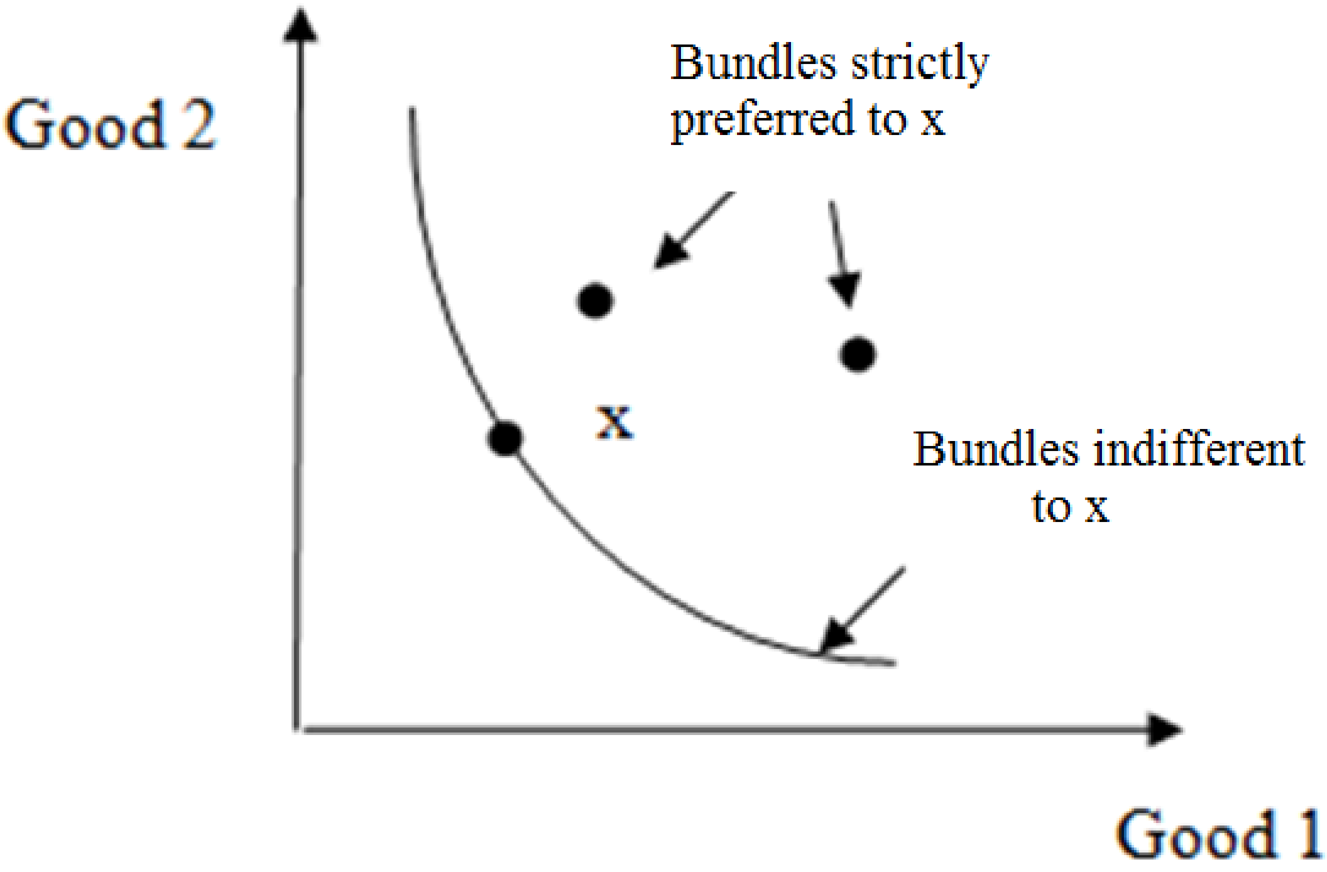
- The set of all consumption bundles that are indifferent to each other is called an indifference curve.

- Points yielding different utility levels are each associated with distinct indifference curves.

Indifference curves are

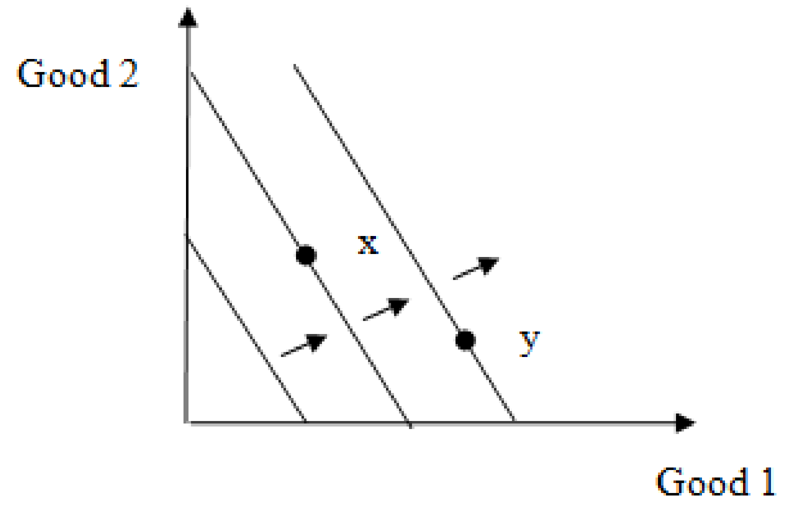
- **Negatively sloped** – as quantity consumed of good 1 increases, total satisfaction would increase if not offset by a decrease in the quantity consumed of the other good 2.
- **Complete**, such that all points on an indifference curve are ranked equally preferred and ranked either more or less preferred than every other point not on the curve. **No two curves can intersect**.
- **Transitive** with respect to points on distinct indifference curve. That is, if each point on indifference curve I_2 is (strictly) preferred to each point on I_1, and each point on I_3 is preferred to each point on I_2, each point on I_3 is preferred to each point on I_1.
- **Convex** – convexity implies that an agent prefers averages to extremes, but, other than that, it has little economic content.

Indifference curve for normal goods



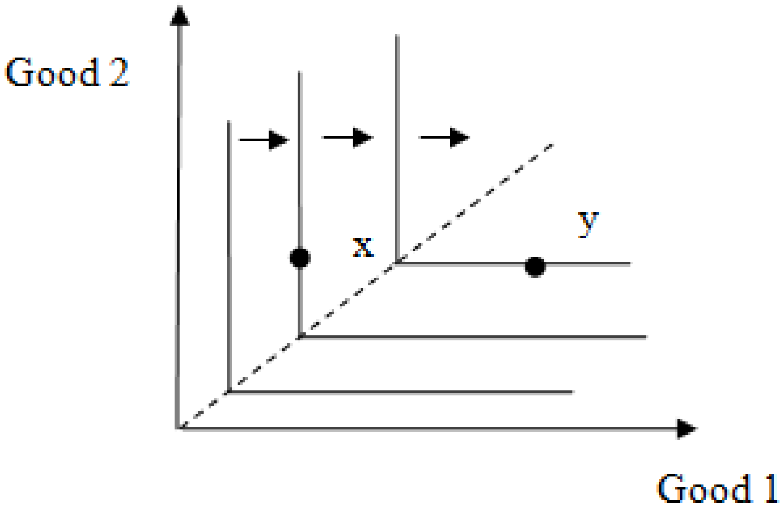
Substitutes

- Two goods are substitutes if the consumer is willing to substitute one good for the other at a constant rate.
- The case of perfect substitutes occurs when the consumer is willing to substitute the goods on a one-to-one basis.
- The indifference curves has a constant slope since the consumer is willing to trade at a fixed ratio.

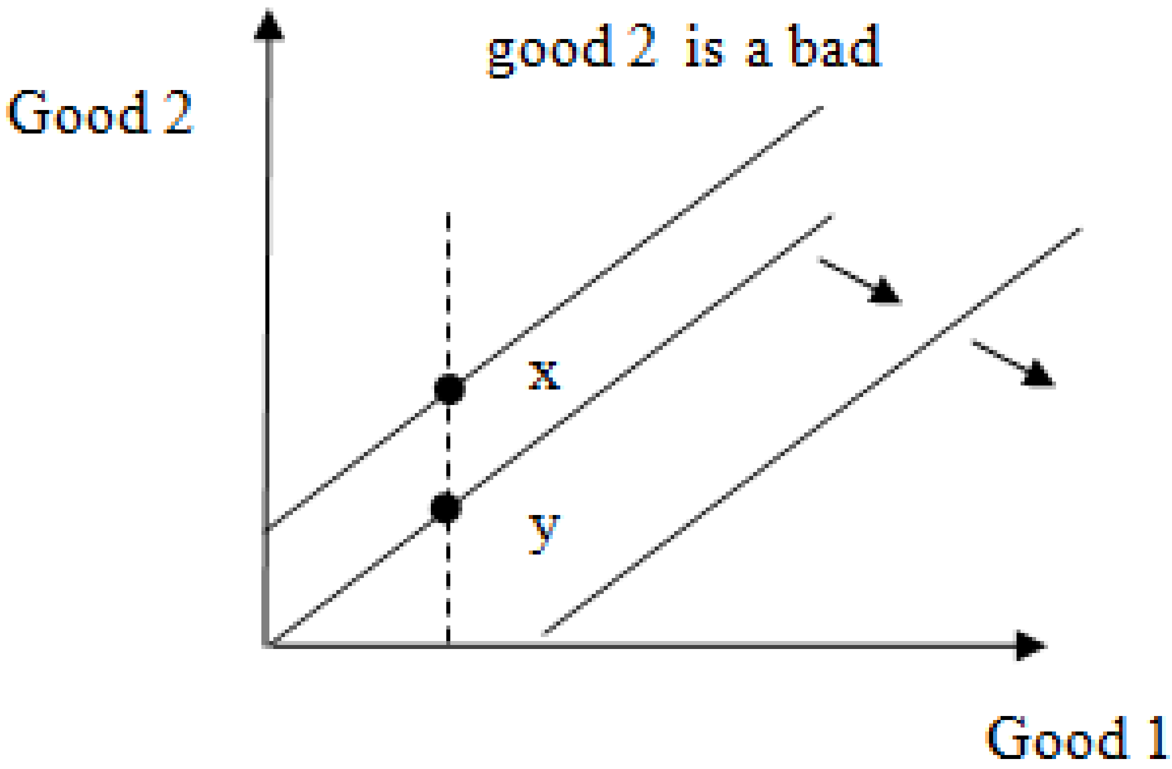


Complements

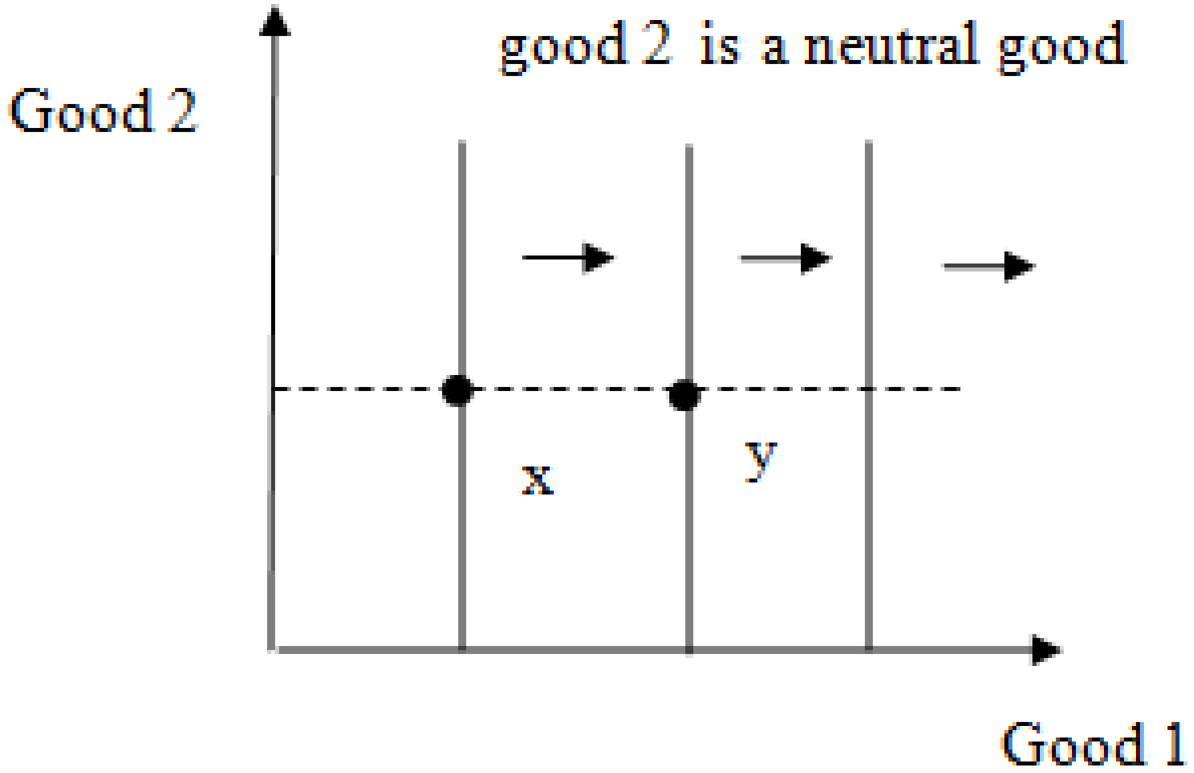
- Complements are goods that are always consumed together in fixed proportions.
- L-shaped indifference curves.



Bads: a bad is a commodity that consumer doesn't like

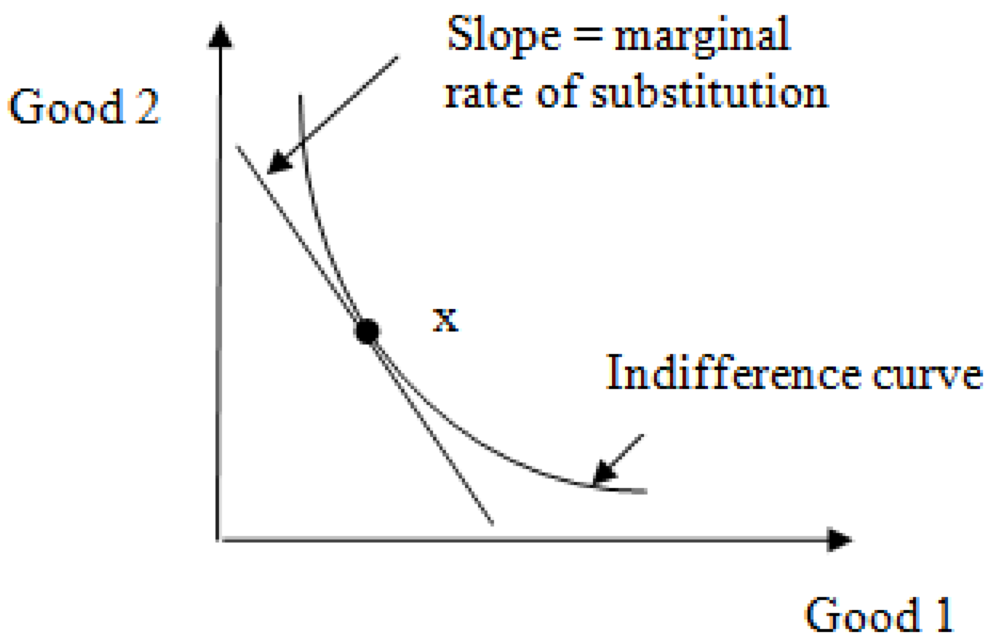


Neutrals: a good is a neutral good if the consumer doesn't care about it one way or the other



The Marginal Rate of Substitution (MRS)

- The marginal rate of substitution measures the slope of the indifference curve.



The Marginal Rate of Substitution (MRS)

$MRS = -\frac{dx_2}{dx_1}$ – the rate at which consumer is ready to give up good 2 in exchange for good 1 while maintaining the same level of satisfaction.

For example $MRS = 3$ – the consumer will give up 3 units of good 2 to obtain 1 additional unit of good 1.

$$dx_1 = 1 \Rightarrow dx_2 = -MRS \cdot dx_1 = -3 \cdot 1 = -3$$

The Marginal Rate of Substitution (MRS)

- The MRS is different at each point along the indifference curve for normal goods.

- The marginal rate of substitution between perfect substitutes is constant.

Mathematical Economics

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Lecture 2

- The Utility Function,
- Examples of Utility Functions: Normal Good, Perfect Substitutes, Perfect Complements,
- The Quasilinear and Homothetic Utility Functions,
- The Marginal Utility and The Marginal Rate of Substitution,
- The Optimal Choice,
- The Utility Maximization Problem,
- The Lagrange Method

The Utility Function

- A utility is a measure of the relative satisfaction from consumption of various goods.

- A utility function is a way of assigning a number to every possible consumption bundle such that more-preferred bundles get assigned larger numbers then less-preferred bundles.

The Utility Function

- The numerical magnitudes of utility levels have no intrinsic meaning – the only property of a utility assignment that is important is how it orders the bundles of goods.

- The magnitude of the utility function is only important insofar as it ranks the different consumption bundles.

- **Ordinal utility** - consumer assigns a higher utility to the chosen bundle than to the rejected. Ordinal utility captures only ranking and not strength of preferences.

- **Cardinal utility** theories attach a significance to the magnitude of utility. The size of the utility difference between two bundles of goods is supposed to have some sort of significance.

Existence of a Utility Function

- Suppose preferences are complete, reflexive, transitive, continuous, and strongly monotonic.
- Then there exists a continuous utility function

$$u: \quad \Re_{+}^{2} \rightarrow \Re$$

which represents those preferences.

The Utility Function

- A utility function is a function *u* assigning a real number to each consumption bundle so that for a pair of bundles *x* and *y*:

$$u(x) > u(y) \iff x \succ y\,,$$
$$u(x) = u(y) \iff x \sim y\,,$$
$$u(x) \geq u(y) \iff x \succsim y\,.$$

Examples of Utility Functions

Goods	The utility function
Normal	$u(x_1, x_2) = x_1^c x_2^d$, $c, d > 0$ - the Cobb-Douglas utility function
Perfect substitutes	$u(x_1, x_2) = a_1 x_1 + a_2 x_2$, $a_1, a_2 > 0$
Perfect complements	$u(x_1, x_2) = \min\{a_1 x_1, a_2 x_2\}$, $a_1, a_2 > 0$

The Quasilinear Utility Function

- The quasilinear (partly linear) utility function is linear in one argument.
- For example the utility function linear in good 2 is the following:

$$u(x_1, x_2) = v(x_1) + x_2$$

The Quasilinear Utility Function

- Specific examples of quasilinear utility would be:

$$u(x_1, x_2) = \sqrt{x_1} + x_2$$

or

$$u(x_1, x_2) = \ln x_1 + x_2$$

The Homothetic Utility Function

A function is called **homothetic** if it is a positive monotonic transformation of a homogenous function.

A function $u: \Re^n \to \Re$ is **homogenous** of degree 1 if $u(\alpha x) = \alpha u(x)$ for all $\alpha > 0$.

A proportional increase in consumption of all goods yields a proportional increase in utility.

Example: $u(x_1, x_2) = \left(x_1^\rho + x_2^\rho\right)^{\frac{1}{\rho}}$ or $u(x_1, x_2) = x_1^c x_2^{1-c}$

The Homothetic Utility Function

A function $u(x)$ is homothetic if $u(x) = g(h(x))$ where g is a strictly increasing function and h is a homogenous of degree 1 function.

So a function is homothetic in x if it can be decomposed into inner function that is homogenous of degree one in x and an outer function monotonically increasing in its argument.

The Homothetic Utility Function

- Slopes of indifference curves are constant along a ray through the origin.

- Assuming that preferences can be represented by a homothetic function is equivalent to assuming that they can be represented by a function that is homogenous of degree 1 because a utility function is unique up to a positive monotonic transformation.

The Marginal Utility

Consider a consumer who is consuming some bundle of goods (x_1, x_2). How does the consumer's utility change as the consumer is given a little more of one of goods.

The marginal utility with respect to good i is a change in utility due to an incremental change in consumption of good i holding consumption of other good fixed: $MU_i = \dfrac{\partial u(x)}{\partial x_i} > 0$, $i = 1, 2$.

The magnitude of marginal utility depends on the magnitude of utility.

The Marginal Rate of Substitution

- Suppose that we increase the amount of good *i*; how does the consumer have to change their consumption of good *j* in order to keep utility constant?

Let dx_i and dx_j be the changes in x_i and x_j ($i, j = 1, 2$, $i \neq j$).

$$\frac{dx_j}{dx_i} = -\frac{\partial u(x)}{\partial x_i} \Big/ \frac{\partial u(x)}{\partial x_j}$$ - the marginal rate of substitution between goods *i* and *j*.

The Marginal Rate of Substitution

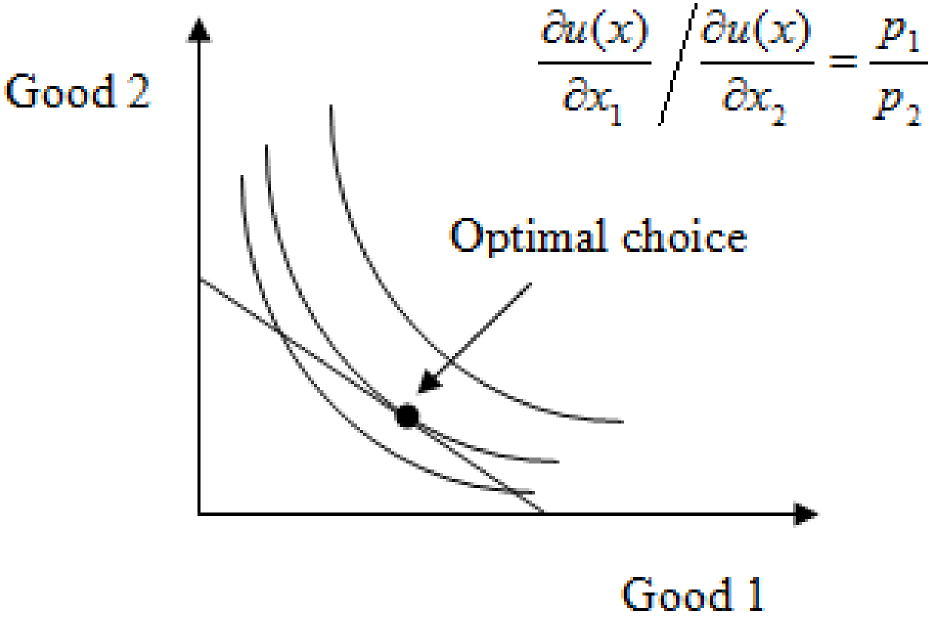
The marginal rate of substitution does not depend on the utility function chosen to represent the underlying preferences.

Let $v(u)$ be a monotonic transformation of utility. The marginal rate of substitution for this utility function is

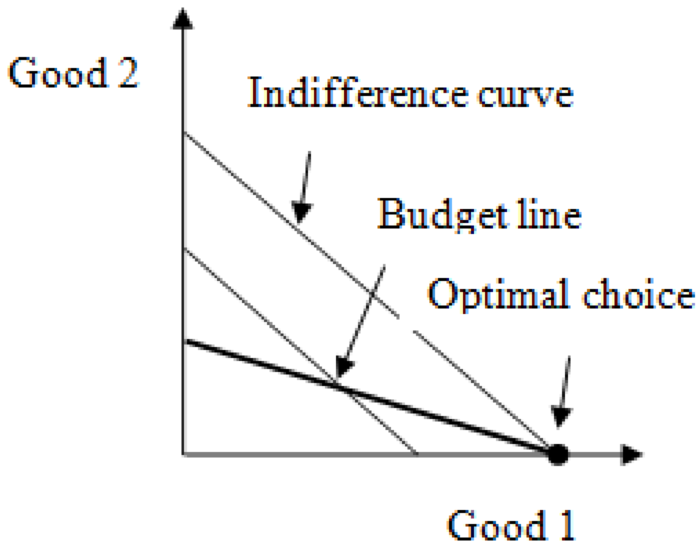
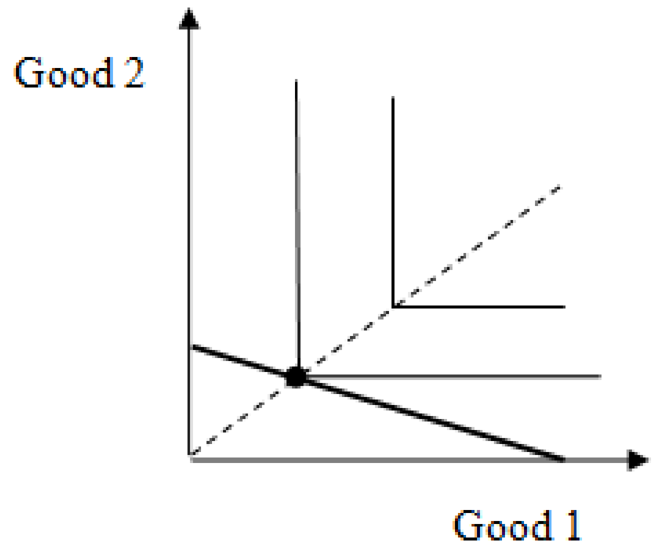
$$\frac{dx_j}{dx_i} = -\left(\frac{dv}{du}\frac{\partial u(x)}{\partial x_i}\right) \Big/ \left(\frac{dv}{du}\frac{\partial u(x)}{\partial x_j}\right) = -\frac{\partial u(x)}{\partial x_i} \Big/ \frac{\partial u(x)}{\partial x_j}.$$

The Optimal Choice

- Consumers choose the most preferred bundle from their budget sets.
- The optimal choice of consumer is that bundle in the consumer's budget set that lies on the highest indifference curve.



The Optimal Choice

Perfect substitutes	**Perfect complements**
	

The Optimal Choice

$$u(x_1, x_2) = \left(\frac{1}{2}x_1 + 2\right)(x_2 + 4)$$

$$2x_1 + x_2 = 8$$

$$\begin{cases} \dfrac{x_2 + 4}{x_1 + 4} = \dfrac{2}{1} \\ 2x_1 + x_2 = 8 \end{cases}$$

$$(1, 6)$$

The Optimal Choice

- Utility functions

a) $u(x_1, x_2) = x_1 + x_2$,
b) $u(x_1, x_2) = 4x_1 + x_2$,

- Budget line

$$2x_1 + x_2 = 8$$

The Optimal Choice

$$u(x_1, x_2) = \min\{2x_1 + x_2, x_1 + 2x_2\}$$

$$2x_1 + 3x_2 = 10$$

$$\begin{cases} x_1 = x_2 \\ 2x_1 + 3x_2 = 10 \end{cases}$$

$$(2, 2)$$

The Utility Maximization

- The problem of utility maximization can be written as:

$$\max_{x_1, x_2} u(x_1, x_2)$$

$$\text{such that}$$

$$p_1 x_1 + p_2 x_2 = I$$

- Consumers seek to maximize utility subject to their budget constraint.

- The consumption levels which solve the utility maximization problem are the Marshallian demand functions.

The Lagrange Method

- The method starts by defining an auxiliary function known as the Lagrangean:

$$L(x_1, x_2, \lambda) = u(x_1, x_2) + \lambda(I - p_1 x_1 - p_2 x_2)$$

- The new variable λ is called a Lagrange multiplier since it is multiplied by constraint.

The Lagrange Method

The Lagrange's theorem says that an optimal choice $(\tilde{x}_1, \tilde{x}_2)$ must satisfy the three first-order conditions:

$$\frac{\partial L}{\partial x_1} = \frac{\partial u(\tilde{x}_1, \tilde{x}_2)}{\partial x_1} - \lambda p_1 = 0,$$

$$\frac{\partial L}{\partial x_2} = \frac{\partial u(\tilde{x}_1, \tilde{x}_2)}{\partial x_2} - \lambda p_2 = 0,$$

$$\frac{\partial L}{\partial \lambda} = \left(I - p_1\tilde{x}_1 - p_2\tilde{x}_2\right) = 0.$$