Mathematical Economics

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Lecture 5

- Neoclassical producer theory. Perfectly competitive firms.
- The profit function and profit maximization problem.
- Properties of the input demand and the output supply.
- Cost minimization problem. Definition and properties of the conditional factor demand and the cost function.
- Profit maximization with the cost function. Long and short run equilibrium.

Perfect Competition

- Perfect competition describes a market in which there are many small firms, all producing homogeneous goods.
- No entry/exit barriers.
- The firm takes prices as a given in both its output and factor markets.
- Perfect information.
- Firms aim to maximize profits.

Profit Maximization in the Long Run

$$\max_{x_1, x_2} \pi(x_1, x_2) = p \cdot f(x_1, x_2) - (v_1 x_1 + v_2 x_2)$$

or

$$\max_{y} \pi(y) = p \cdot y - c(y)$$

where $c(y)$ is the cost function.

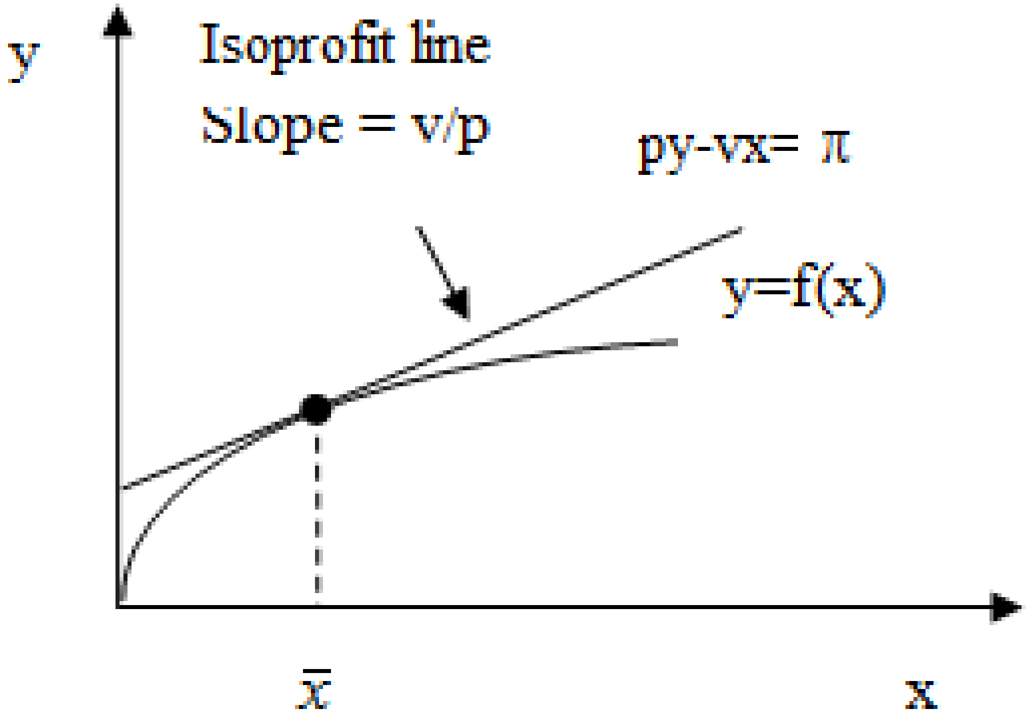
$$\max_{x_1, x_2} \pi(x_1, x_2) = p \cdot f(x_1, x_2) - (v_1 x_1 + v_2 x_2)$$

The first-order conditions for the profit maximization problem are

$$\begin{cases} \dfrac{\partial \pi(x_1, x_2)}{\partial x_1} = 0 \\ \dfrac{\partial \pi(x_1, x_2)}{\partial x_2} = 0 \end{cases} \Rightarrow \begin{cases} p \dfrac{\partial f(x_1, x_2)}{\partial x_1} = v_1 \\ p \dfrac{\partial f(x_1, x_2)}{\partial x_2} = v_2 \end{cases} \Rightarrow \quad \bar{x} = (\bar{x}_1, \bar{x}_2)$$

The solution of firm's profit maximization problem $\bar{x} = (\bar{x}_1, \bar{x}_2)$ is the factor demand function.

The first-order condition can be exhibited graphically



The tangency condition $\frac{df(x)}{dx} = \frac{v}{p}$

The Factor Demand Function $\bar{x}(p, v)$

- The factor demand function $\bar{x}(p, v)$ gives the optimal choice of inputs as a function of the prices.
- Homogenous of degree 0 in (p, v).
- Decreasing in factor prices.

The Supply Function $\bar{y}(p, v)$

- The supply function of the firm $\bar{y} = f(\bar{x}_1, \bar{x}_2)$.
- Homogenous of degree 0 in (p, v).
- Increasing in output price.

The Profit Function $\pi(p, v)$

- The profit function $\pi(\bar{x}_1, \bar{x}_2) = \pi(p, v)$
- Nondecreasing in output price p, nonincreasing in input prices (v_1, v_2).
- Homogenous of degree 1 in (p, v).
- Continuous in (p, v).
- Convex in (p, v).

Cost Minimization in the Long Run

$$\min_{x_1, x_2} v_1 x_1 + v_2 x_2$$

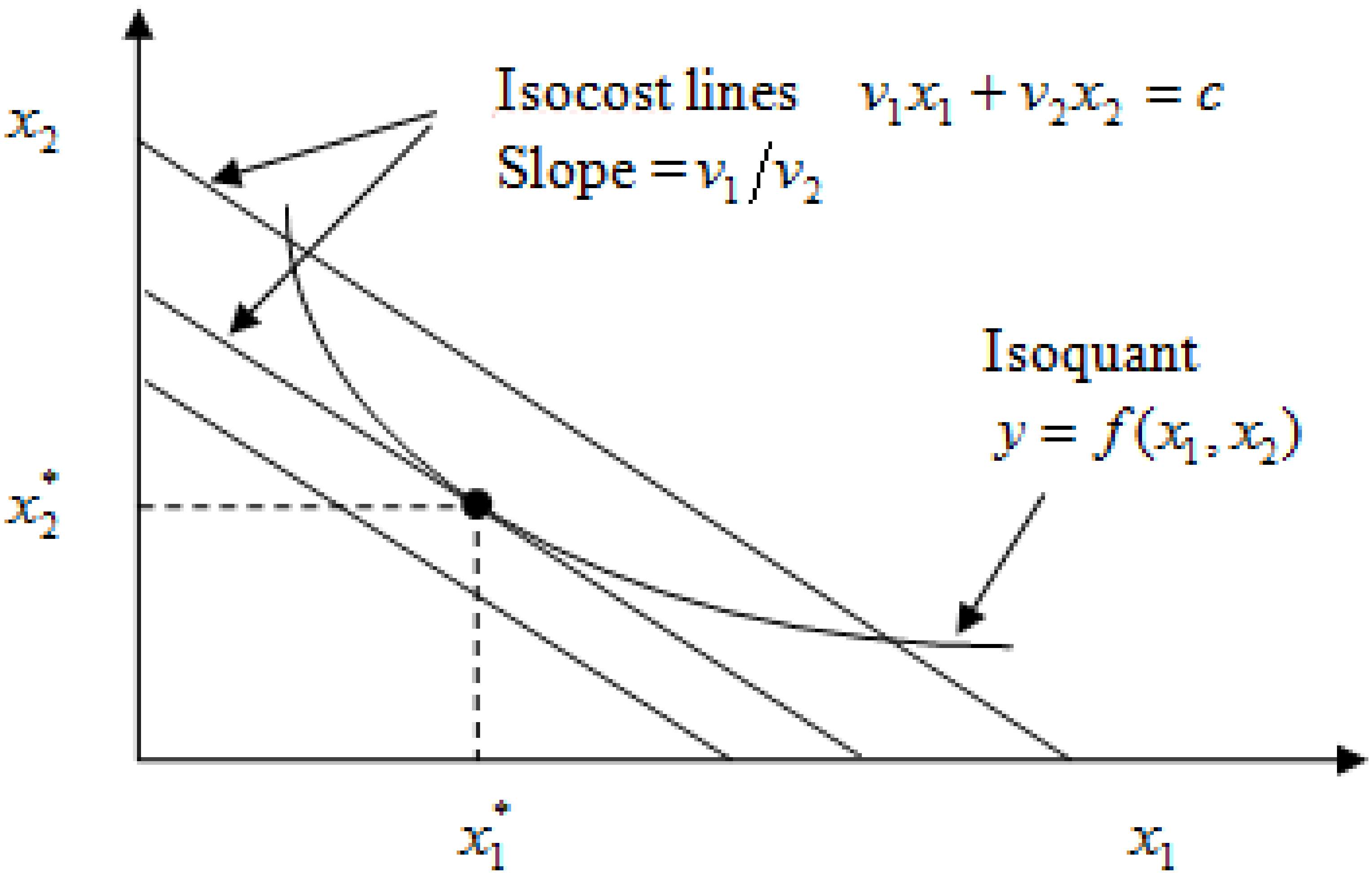
such that

$$y = f(x_1, x_2)$$

$$L(x_1, x_2, \lambda) = v_1 x_1 + v_2 x_2 + \lambda(y - f(x_1, x_2))$$

$$\begin{cases} \dfrac{\partial L(x_1, x_2, \lambda)}{\partial x_1} = 0 \\ \dfrac{\partial L(x_1, x_2, \lambda)}{\partial x_2} = 0 \\ \dfrac{\partial L(x_1, x_2, \lambda)}{\partial \lambda} = 0 \end{cases} \Rightarrow \begin{cases} v_1 = \lambda \dfrac{\partial f(x_1, x_2)}{\partial x_1} \\ v_2 = \lambda \dfrac{\partial f(x_1, x_2)}{\partial x_2} \\ y = f(x_1, x_2) \end{cases}$$

The cost minimization problem can be exhibited graphically



The Conditional Factor Demand Function $x^*(v, y)$

The level of each input that solves the cost minimization problem are called conditional input demands and are denoted

$$x^*(v, y) = \left(x_1^*(v, y), x_2^*(v, y)\right)$$

The Cost Function $c(v, y)$

- Inserting x_1^* and x_2^* into the total cost expression yields the cost function $c(v, y) = v_1 x_1^* + v_2 x_2^*$.
- $c(v, y)$ measures the minimum costs of producing a given level of output at given factor prices.
- $c(v, y)$ is increasing in y and nondecreasing in factor prices.
- $c(v, y)$ is homogenous of degree 1 in v.
- $c(v, y)$ is concave in v.

Profit Maximization with the Cost Function

$$\max_{y} \; \pi(y) = p \cdot y - c(y)$$

$$\frac{d\pi(y)}{dy} = 0 \quad \Rightarrow \quad p = \frac{dc(y)}{dy} \quad \Rightarrow \quad \bar{y}$$

Example 1

Derive the factor demand function, the supply function, the profit function, the conditional factor demand function and the cost function of the firm with the Cobb-Douglas production technology $y = 4x_1^{\frac{1}{4}} x_2^{\frac{1}{2}}$.

Solve profit maximization problem with the cost function.

$$\max_{x_1, x_2} \pi(x_1, x_2) = 4px_1^{\frac{1}{4}} x_2^{\frac{1}{2}} - (v_1 x_1 + v_2 x_2)$$

- $\bar{x}_1 = \dfrac{4p^4}{v_1^2 v_2^2}$, $\bar{x}_2 = \dfrac{8p^4}{v_1 v_2^3}$ - the factor demand function,
- $\bar{y} = \dfrac{16p^3}{v_1 v_2^2}$ - the supply function,
- $\pi(p, v) = \dfrac{4p^4}{v_1 v_2^2}$ - the profit function,

$$\min_{x_1,x_2} \; v_1 x_1 + v_2 x_2$$

such that

$$y = 4 x_1^{\frac{1}{4}} x_2^{\frac{1}{2}}$$

- $x_1^* = \left(\frac{v_2}{2v_1}\right)^{\frac{2}{3}} \left(\frac{y}{4}\right)^{\frac{4}{3}}, \quad x_2^* = \left(\frac{2v_1}{v_2}\right)^{\frac{1}{3}} \left(\frac{y}{4}\right)^{\frac{4}{3}}$

  - the conditional factor demand function

- $c(v, y) = 3 \cdot 2^{-\frac{2}{3}} \cdot v_1^{\frac{1}{3}} \cdot v_2^{\frac{2}{3}} \cdot \left(\frac{y}{4}\right)^{\frac{4}{3}}$

- the cost function

$$\max_{y}\ \pi(y) = p \cdot y - 3 \cdot 2^{-\frac{2}{3}} \cdot v_1^{\frac{1}{3}} \cdot v_2^{\frac{2}{3}} \cdot \left(\frac{y}{4}\right)^{\frac{4}{3}}$$

- $\bar{y} = \dfrac{16 p^3}{v_1 v_2^2}$.

The cost function for

- The Leontief technology

$$f(x_1, x_2) = \min\{a_1 x_1, a_2 x_2\} \text{ is given by } c(v, y) = \left(\frac{v_1}{a_1} + \frac{v_2}{a_2}\right) \cdot y$$

- The linear technology

$$f(x_1, x_2) = a_1 x_1 + a_2 x_2 \text{ is given by } c(v, y) = \min\left\{\frac{v_1}{a_1}, \frac{v_2}{a_2}\right\} \cdot y$$

Profit Maximization and Cost Minimization in the Short Run

Suppose that in the short run factor 1 is fixed at some predetermined level $\hat{x}_1$.

$y = f(\hat{x}_1, x_2)$ - the production function in short run.

1. $\max_{x_2} \pi(\hat{x}_1, x_2) = p \cdot f(\hat{x}_1, x_2) - (v_1\hat{x}_1 + v_2 x_2)$

$$\frac{\partial \pi(\hat{x}_1, x_2)}{\partial x_2} = 0 \quad \Rightarrow \quad p\frac{\partial f(\hat{x}_1, x_2)}{\partial x_2} = v_2 \quad \Rightarrow \quad \bar{x}_2$$

2. $\min_{x_2} v_1\hat{x}_1 + v_2 x_2$ such that $y = f(\hat{x}_1, x_2)$ $\quad \Rightarrow \quad x_2^* = x_2^*(v, y, \hat{x}_1)$

$$c(v, y) = v_1\hat{x}_1^* + v_2 x_2^*$$

- Total Costs, Average Costs,
- Marginal Costs,
- Long-run Costs, Short-run Costs,
- Cost Curves, Long-run and Short-run Cost Curves,
- Monopoly

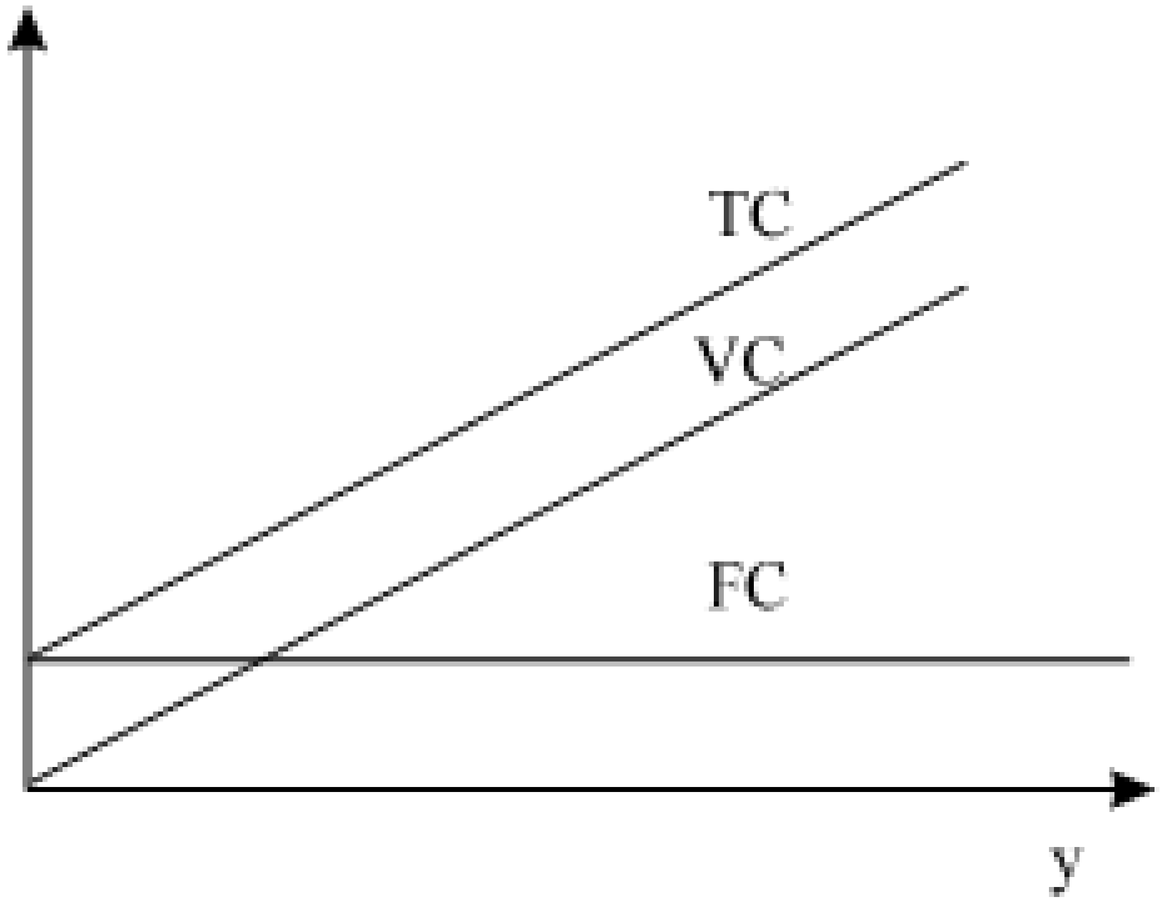
Total Costs

total costs (TC) = variable costs (VC) + fixed costs (FC)

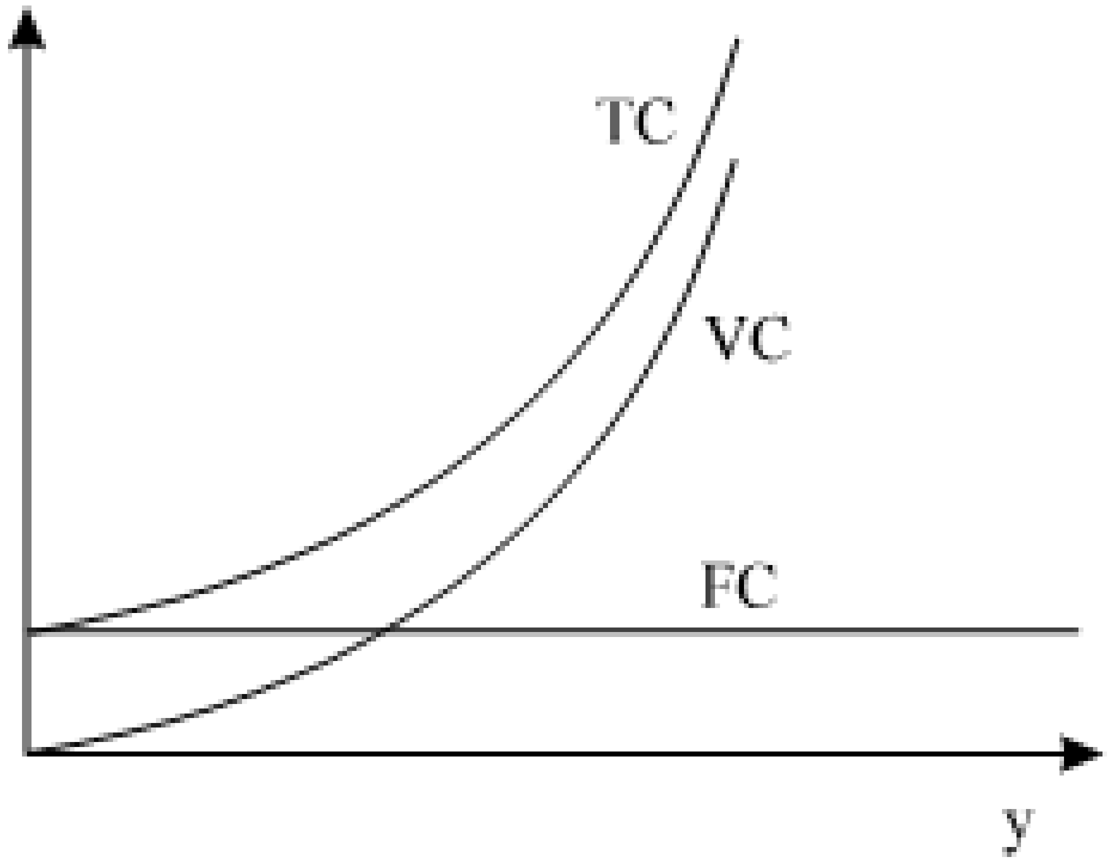
$$TC = VC + FC \qquad (TC \equiv c(y))$$

$$FC \equiv TC(0)$$

$TC = \alpha y + \beta$, $VC = \alpha y$, $FC = \beta$, $\alpha, \beta > 0$

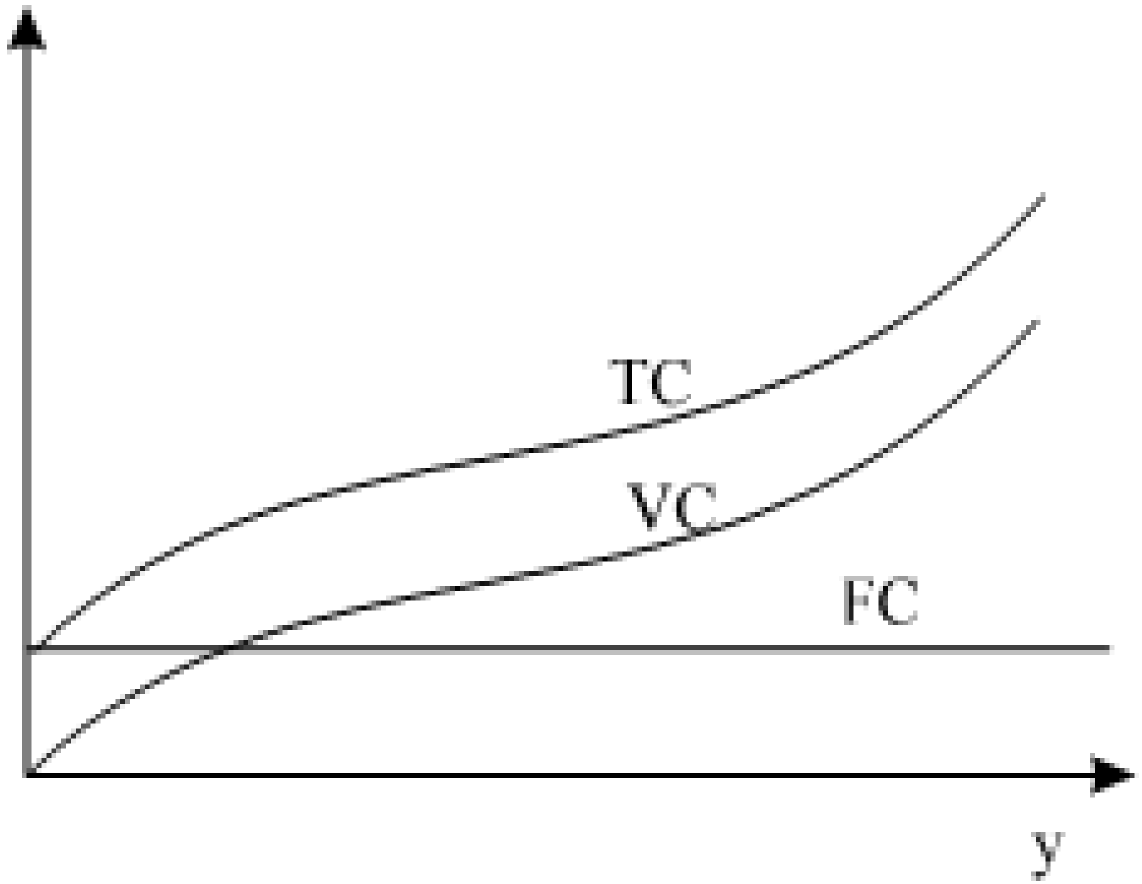


$TC = \alpha y^2 + \beta y + \gamma$, $\quad VC = \alpha y^2 + \beta y$, $\quad FC = \gamma$, $\quad \alpha, \beta, \gamma > 0$

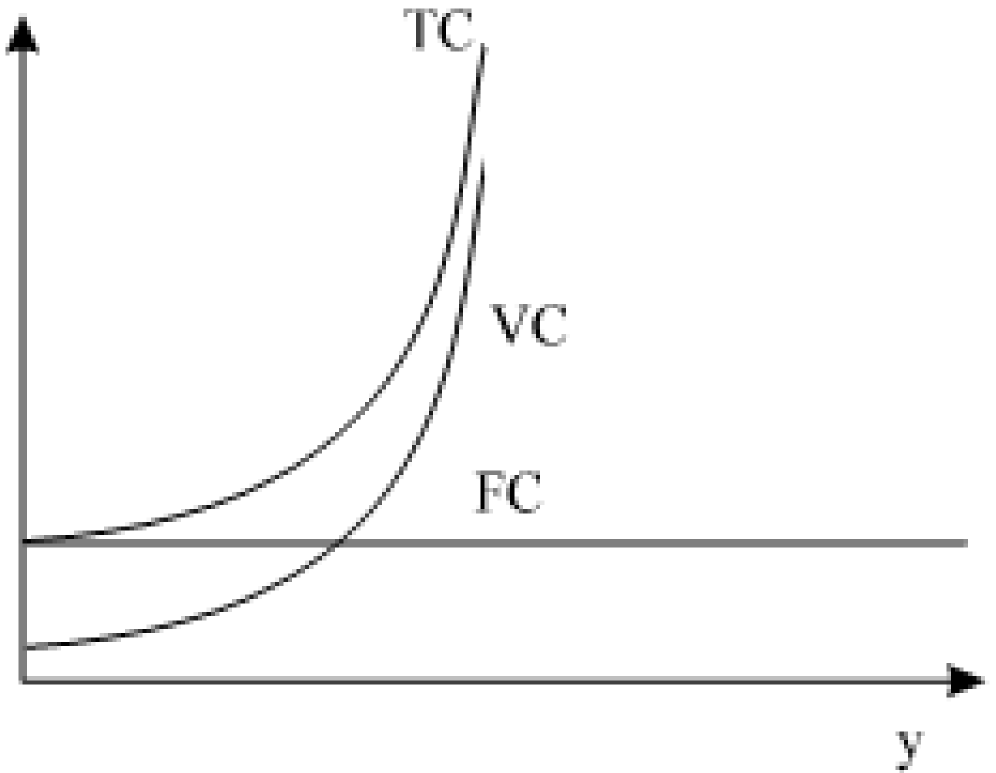


$TC = \alpha y^3 + \beta y^2 + \gamma y + \delta$，　　$VC = \alpha y^3 + \beta y^2 + \gamma y$，　　$FC = \delta$

$\alpha, \gamma, \delta > 0$，　　$\beta < 0$，　$\beta^2 < 3\alpha\gamma$

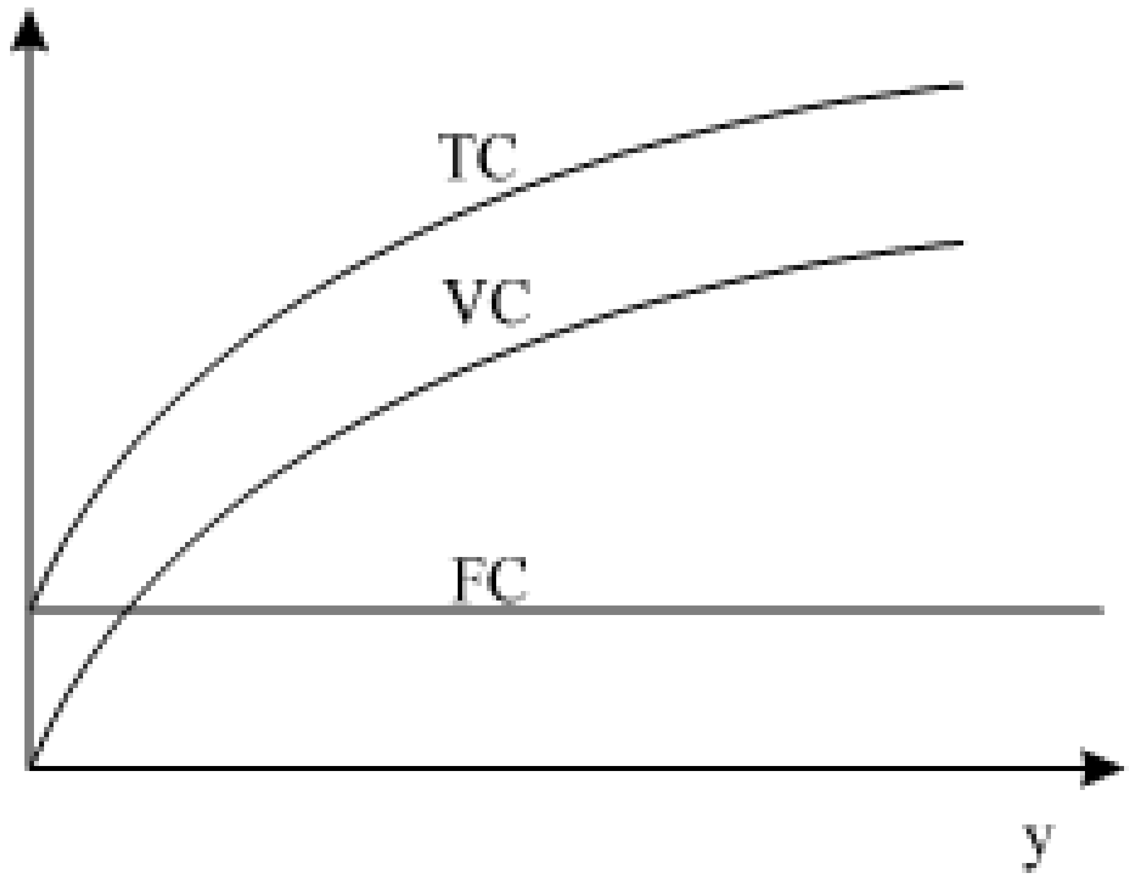


$TC(y) = \alpha e^{\beta y}$, $VC(y) = \alpha e^{\beta y}$, $FC = \alpha$, $\alpha, \beta > 0$



Note: $FC = TC(0) = \alpha e^{\beta y} = \alpha e^{\beta \cdot 0} = \alpha \cdot 1$

$TC(y) = \alpha y^{\beta} + \gamma$, $VC(y) = \alpha y^{\beta}$, $FC = \gamma$, $\alpha, \gamma > 0$, $0 < \beta < 1$



Average Costs

- The average cost function measures the cost per unit of output

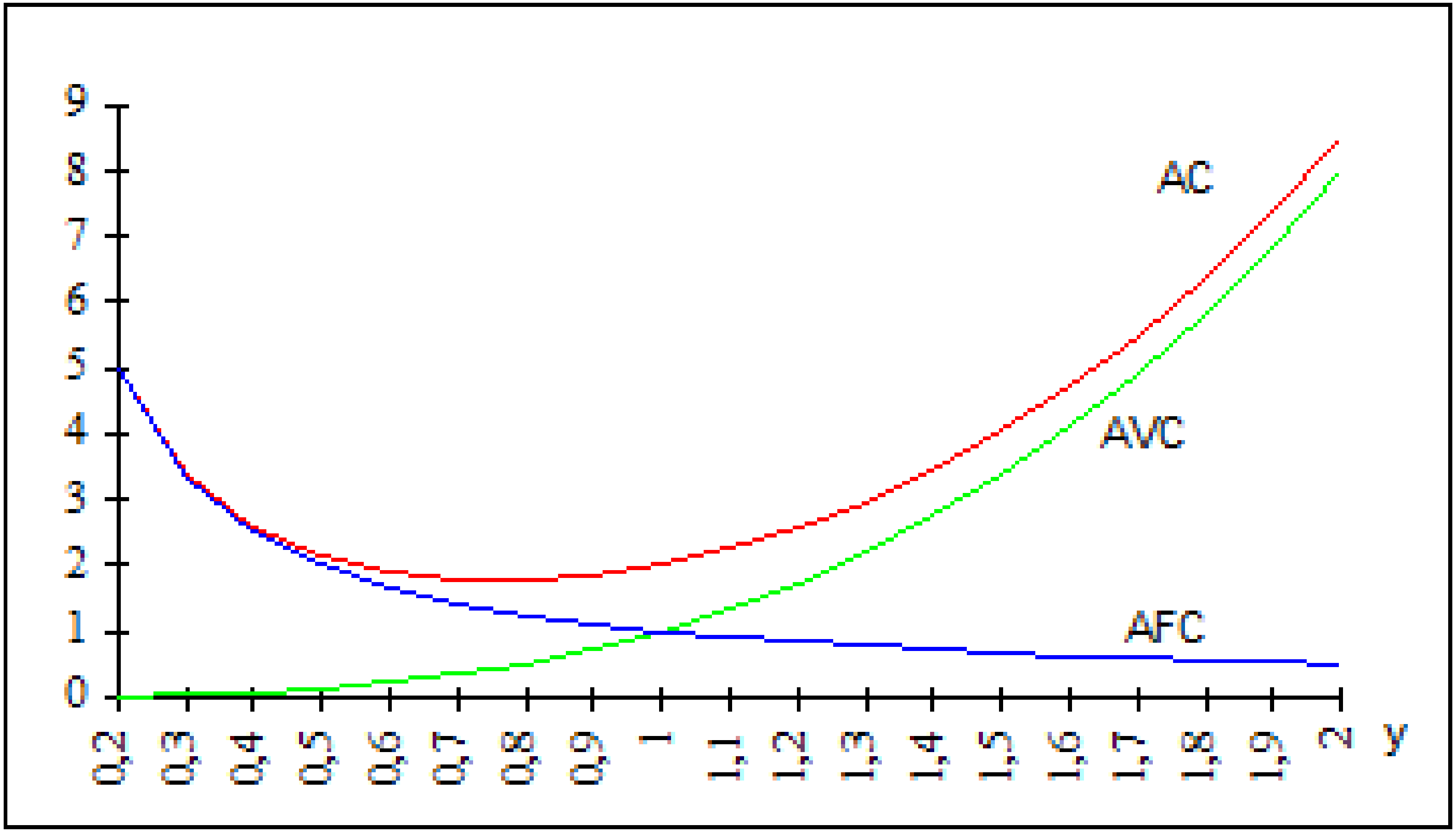
$$AC(y) = \frac{TC(y)}{y}$$

- Average costs (AC) = average variable costs (AVC) + average fixed costs (AFC)

$$AC(y) = \frac{VC(y)}{y} + \frac{FC(y)}{y} = AVC(y) + AFC(y)$$

- Average fixed costs always decline with output, while average variable costs tend to increase. The net result is a U-shaped average cost curve.

$$TC = y^4 + 1, \quad AC = y^3 + \frac{1}{y}, \qquad AVC = y^3, \quad AFC = \frac{1}{y}$$



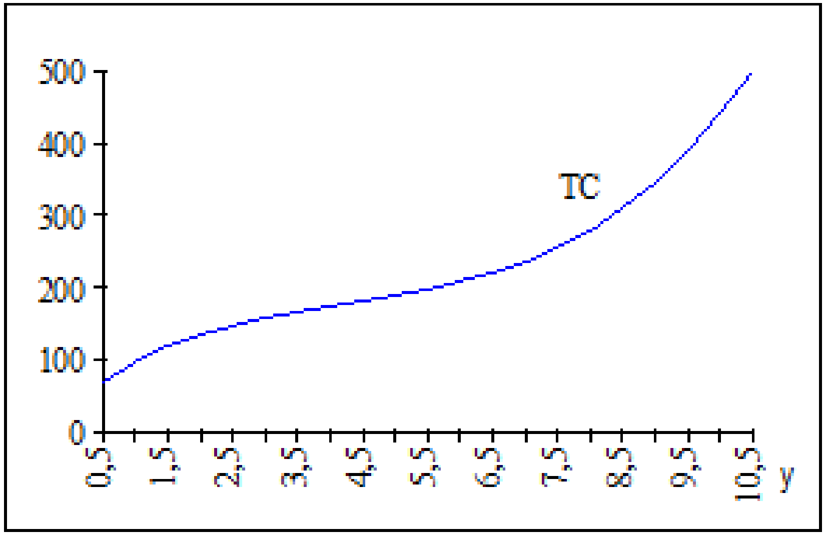
Marginal Costs

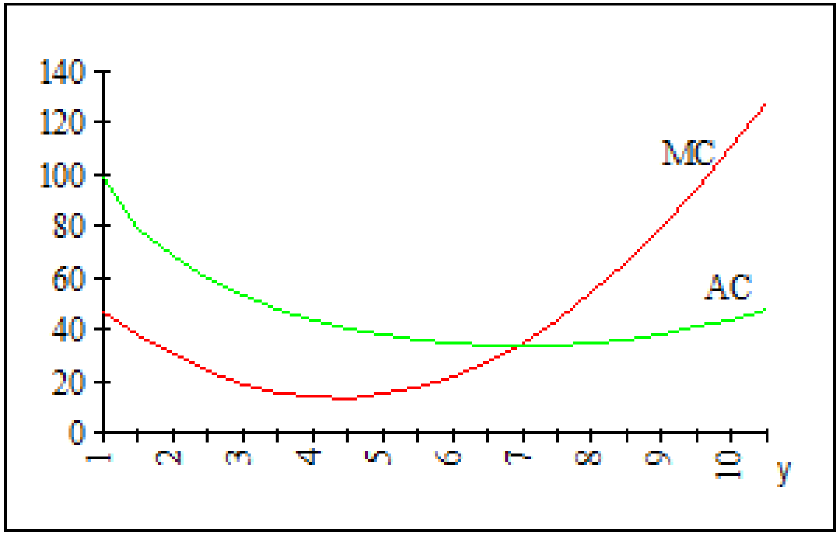
$$MC = \frac{dTC(y)}{dy}$$

- The marginal cost curve lies below the average cost curve when average cost is decreasing, and above when they are increasing.
- The marginal costs are equal average costs at the point of minimum average costs.

$$TC(y) = y^3 - 13y^2 + 70y + 40, \qquad AC(y) = y^2 - 13y + 70 + \frac{40}{y},$$

$$MC(y) = 3y^2 - 26y + 70$$





Long-run Average Costs and Short-run Average Costs

$$\min_{x_1,x_2}\left(v_1x_1+v_2x_2\right)$$

such that

$$y = Ax_1^c x_2^d, \quad A,c,d>0$$

$$x_1^* = A^{-\frac{1}{c+d}}\left(\frac{cv_2}{dv_1}\right)^{\frac{d}{c+d}} y^{\frac{1}{c+d}}, \qquad x_2^* = A^{-\frac{1}{c+d}}\left(\frac{cv_2}{dv_1}\right)^{-\frac{c}{c+d}} y^{\frac{1}{c+d}}$$

$$TC(y)\equiv c(y) = K_L\, y^{\frac{1}{c+d}}$$

$$K_L = A^{-\frac{1}{c+d}}\left\{\left(\frac{c}{d}\right)^{\frac{d}{c+d}}+\left(\frac{d}{c}\right)^{\frac{c}{c+d}}\right\} v_1^{\frac{c}{c+d}} v_2^{\frac{d}{c+d}}$$

Long-run Average Costs and Short-run Average Costs

$\hat{x}_2$ - the fixed level of input 2

$$c(y) = \min_{x_1} \left(v_1 x_1 + v_2 \hat{x}_2 \right)$$

such that

$$y = A x_1^c \hat{x}_2^d$$

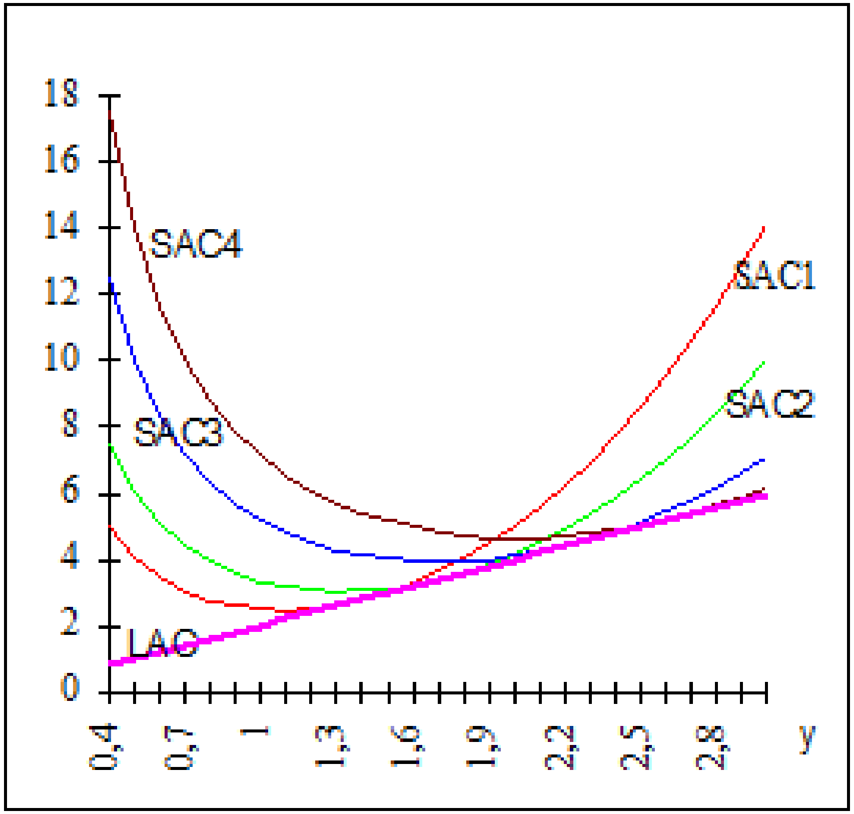
$$x_1^* = A^{-\frac{1}{c}} y^{\frac{1}{c}} \hat{x}_2^{-\frac{d}{c}}$$

$$c(y) = K_S y^{\frac{1}{c}} + F_S \qquad K_S = A^{-\frac{1}{c}} v_1 \hat{x}_2^{-\frac{d}{c}}, \qquad F_S = v_2 \hat{x}_2$$

$$LAC = K_L y^{\frac{1-c-d}{c+d}} \qquad LMC = \frac{K_L}{c+d} y^{\frac{1-c-d}{c+d}}$$

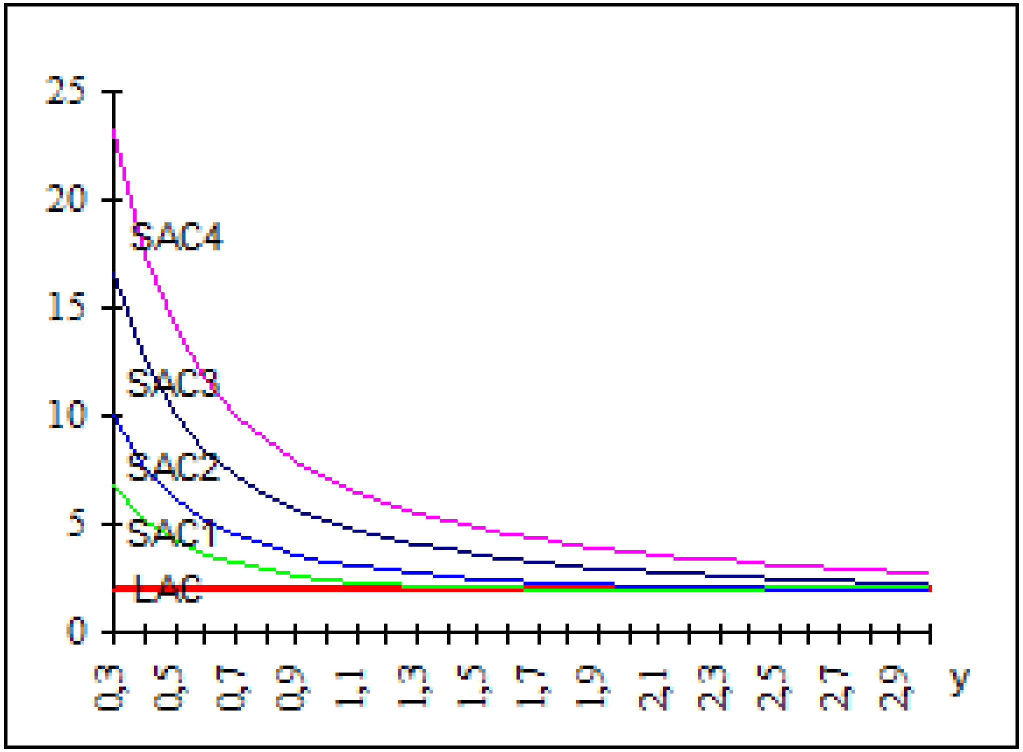
$$SAC = K_S y^{\frac{1-c}{c}} + \frac{F_S}{y} \qquad SMC = \frac{K_S}{c} y^{\frac{1-c}{c}}$$

$$y = x_1^{\frac{1}{4}} x_2^{\frac{1}{4}} \qquad LAC = 2y, \qquad SAC = \frac{y^3}{\hat{x}_2} + \frac{\hat{x}_2}{y}$$



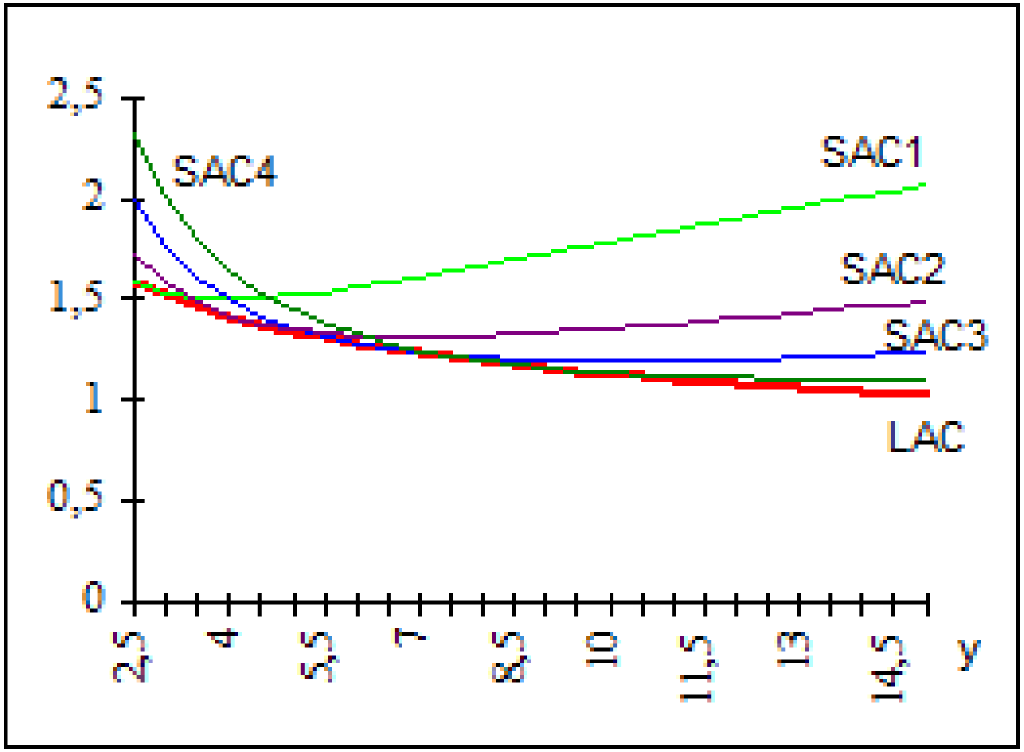
$SAC1: \quad \hat{x}_2 = 2$, $SAC2: \quad \hat{x}_2 = 3$, $\quad SAC3: \quad \hat{x}_2 = 5$, $\qquad SAC4: \quad \hat{x}_2 = 7$

$$y = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}, \qquad LAC = 2, \qquad SAC = \frac{y}{\hat{x}_2} + \frac{\hat{x}_2}{y}$$



$SAC1: \quad \hat{x}_2 = 2$, $SAC2: \quad \hat{x}_2 = 3$, $\quad SAC3: \quad \hat{x}_2 = 5$, $\quad SAC4: \quad \hat{x}_2 = 7$

$$y = x_1^{\frac{2}{3}} x_2^{\frac{2}{3}}, \qquad LAC = 2y^{-\frac{1}{4}}, \qquad SAC = \frac{\sqrt{y}}{\hat{x}_2} + \frac{\hat{x}_2}{y},$$



$SAC1: \quad \hat{x}_2 = 2$, $SAC2: \quad \hat{x}_2 = 3$, $\quad SAC3: \quad \hat{x}_2 = 4$, $\quad SAC4: \quad \hat{x}_2 = 5$

Monopoly

- Monopoly is a price-maker.
- A monopolist has market power in the sense that amount of output that is able to sell responds continuously as a function of the price it charges.

The monopolist's profit maximization problem can be posed as

$$\max_{x_1, x_2} \pi(x_1, x_2) = r(x_1, x_2) - c(x_1, x_2)$$

where $r(x_1, x_2) = p(f(x_1, x_2)) \cdot f(x_1, x_2)$;

$$c(x_1, x_2) = v_1(x_1) \cdot x_1 + v_2(x_2) \cdot x_2$$

The first-order conditions for the profit maximization problem are

$$\begin{cases} \dfrac{\partial \pi(x_1, x_2)}{\partial x_1} = 0 \\ \dfrac{\partial \pi(x_1, x_2)}{\partial x_2} = 0 \end{cases} \Rightarrow \begin{cases} \dfrac{\partial r(x_1, x_2)}{\partial x_1} = \dfrac{\partial c(x_1, x_2)}{\partial x_1} \\ \dfrac{\partial r(x_1, x_2)}{\partial x_2} = \dfrac{\partial c(x_1, x_2)}{\partial x_2} \end{cases}$$

$$\bar{x} = (\bar{x}_1, \bar{x}_2)$$

The monopolist's profit maximization problem can be posed as

$$\max_{y} \pi(y) = r(y) - c(y) = p(y) \cdot y - c(y)$$

where $r(y) = p(y) \cdot y$ is the revenue function,

$c(y)$ is the cost function.

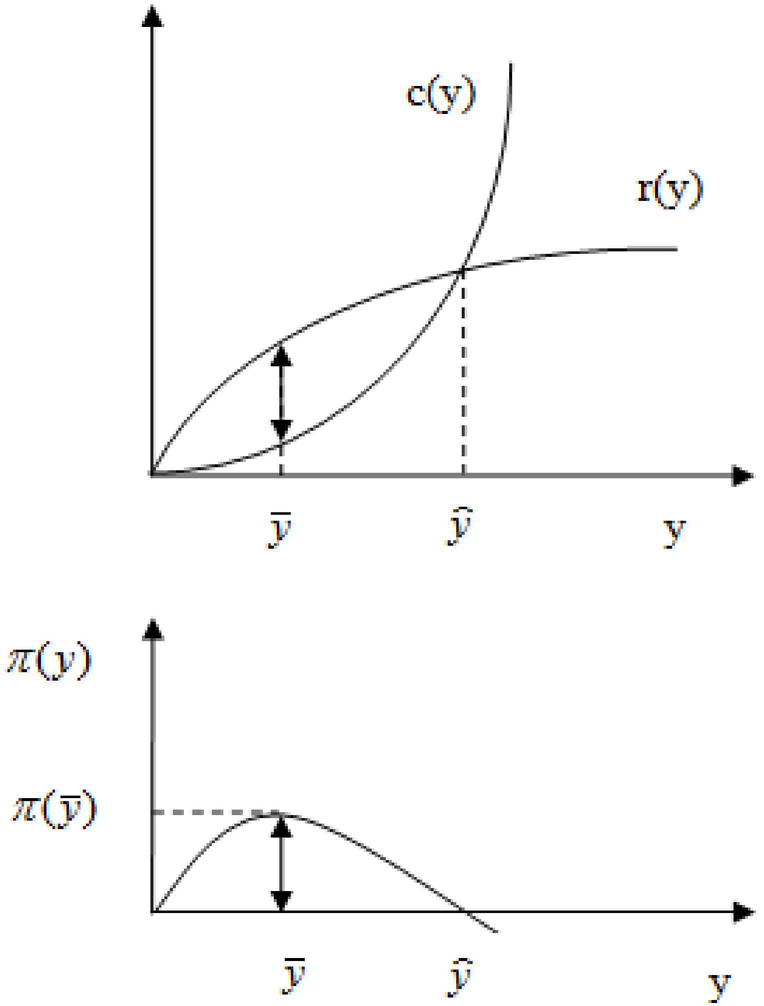
The first-order condition for the profit maximization problem is

$$\frac{d\pi(y)}{dy} = 0 \quad \Rightarrow \quad \frac{dr(y)}{dy} = \frac{dc(y)}{dy}$$

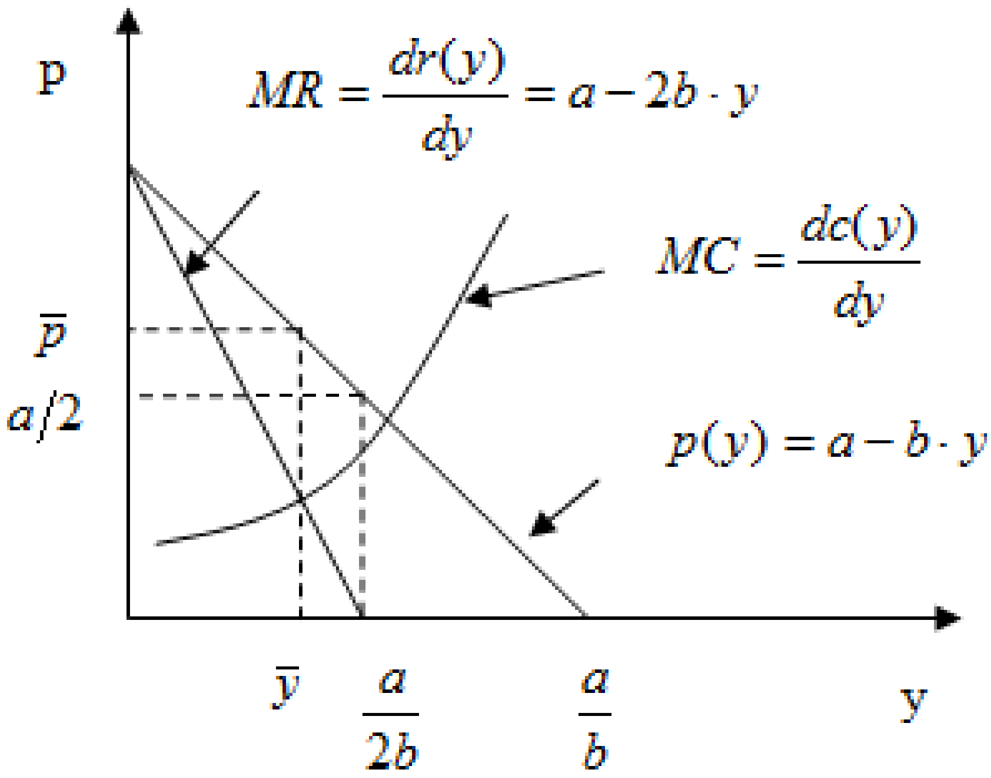
$$\Rightarrow \quad p(y)\left[1 - \frac{1}{|\varepsilon(y)|}\right] = \frac{dc(y)}{dy},$$

where $\varepsilon(y) = \frac{dy}{dp}\frac{p}{y}$ - the price elasticity of demand facing the monopolist.

Monopoly with a Nonlinear Demand Function



Monopoly with a Linear Demand Function



$$\varepsilon(y) = -\frac{p}{a-p} \quad \Rightarrow \quad |\varepsilon(y)| = 1 \quad \Rightarrow \quad y = \frac{a}{2b}, \quad p = \frac{a}{2}$$

Example

Let assume that $y = x_1^{\frac{1}{3}} x_2^{\frac{1}{3}}$, $p(y) = 6y^{-\frac{1}{2}}$, v_1 , v_2

$$\max_{x_1, x_2} \pi(x_1, x_2) = 6x_1^{\frac{1}{6}} x_2^{\frac{1}{6}} - (v_1 x_1 + v_2 x_2)$$

- $\bar{x}_1 = v_1^{-\frac{5}{4}} v_2^{-\frac{1}{4}}$, $\bar{x}_2 = v_1^{-\frac{1}{4}} v_2^{-\frac{5}{4}}$
- $\bar{y} = v_1^{-\frac{1}{2}} v_2^{-\frac{1}{2}}$

$$\min_{x_1, x_2} \; v_1 x_1 + v_2 x_2$$

such that

$$y = x_1^{\frac{1}{3}} x_2^{\frac{1}{3}}$$

- $x_1^* = \left(\dfrac{v_2}{v_1}\right)^{\frac{1}{2}} y^{\frac{3}{2}}, \qquad x_2^* = \left(\dfrac{v_1}{v_2}\right)^{\frac{1}{2}} y^{\frac{3}{2}}$
- $c(y) = 2 \cdot v_1^{\frac{1}{2}} \cdot v_2^{\frac{1}{2}} \cdot y^{\frac{3}{2}}$

$$\max_{y} \pi(y) = 6y^{\frac{1}{2}} - 2 \cdot v_1^{\frac{1}{2}} \cdot v_2^{\frac{1}{2}} \cdot y^{\frac{3}{2}}$$

- $\bar{y} = v_1^{-\frac{1}{2}} v_2^{-\frac{1}{2}}$

Inefficiency of Monopoly