

# Mathematical Economics

dr Wioletta Nowak

Lecture 6

- Oligopoly
- Cournot equilibrium
- Quantity leadership – Stackelberg model

- Exchange
- Market Equilibrium

## Cournot equilibrium

Consider two firms which produce a homogenous product with output levels  $y_1$  and  $y_2$ , and an aggregate output  $Y = y_1 + y_2$ .

The market price associated with this output is taken to be  $p(Y) \equiv p(y_1 + y_2)$ .

Firm  $i$  has a cost function given by  $c_i(y_i)$  for  $i = 1, 2$ .

Firm 1's maximization problem is

$$\max_{y_1} \pi_1(y_1, y_2) = p(y_1 + y_2)y_1 - c_1(y_1)$$

Firm 2's maximization problem is

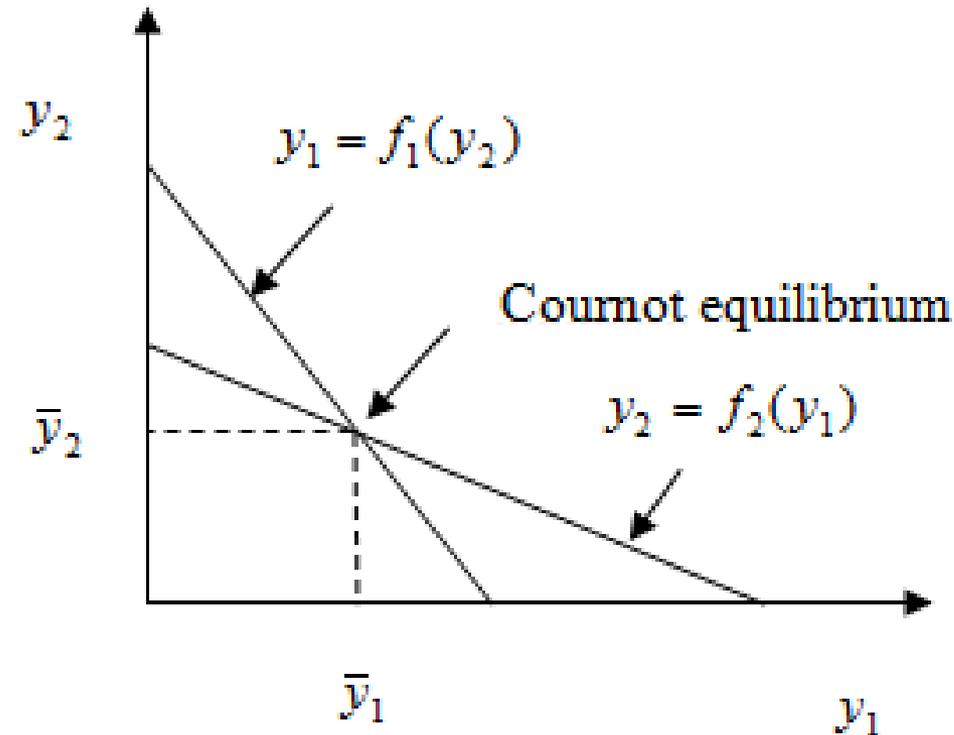
$$\max_{y_2} \pi_2(y_1, y_2) = p(y_1 + y_2)y_2 - c_2(y_2)$$

The first-order conditions for the profit maximization problems are:

$$\begin{cases} \frac{\partial \pi_1(y_1, y_2)}{\partial y_1} = 0 \\ \frac{\partial \pi_2(y_1, y_2)}{\partial y_2} = 0 \end{cases}$$

$$\begin{cases} \frac{\partial p(y_1 + y_2)}{\partial y_1} y_1 + p(y_1 + y_2) - \frac{\partial c_1(y_1)}{\partial y_1} = 0 \\ \frac{\partial p(y_1 + y_2)}{\partial y_2} y_2 + p(y_1 + y_2) - \frac{\partial c_2(y_2)}{\partial y_2} = 0 \end{cases} \Rightarrow (\bar{y}_1, \bar{y}_2)$$

- The first-order condition for firm 1 determines firm 1's optimal choice of output as a function of its beliefs about firm 2's output choice. This relation is known as firm 1's reaction curve. It depicts how firm 1 will react given various beliefs it might have about firm 2's choice.



## Quantity leadership – Stackelberg model

This is a two-stage model in which one firm gets to move first.

The other firm can then observe the first firm's output choice and choose its own optimal level of output.

Suppose that firm 1 is a leader and that it chooses to produce a quantity  $y_1$ . Firm 2 responds by choosing a quantity  $y_2$ .

Each firm knows that the equilibrium price in the market depends on the total output produced.

## The follower's problem

$$\max_{y_2} \pi_2(y_1, y_2) = p(y_1 + y_2)y_2 - c_2(y_2)$$

$$\frac{\partial \pi_2(y_1, y_2)}{\partial y_2} = 0 \quad \Rightarrow \quad y_2 = f_2(y_1) \quad (\text{a reaction function})$$

## The leader's problem

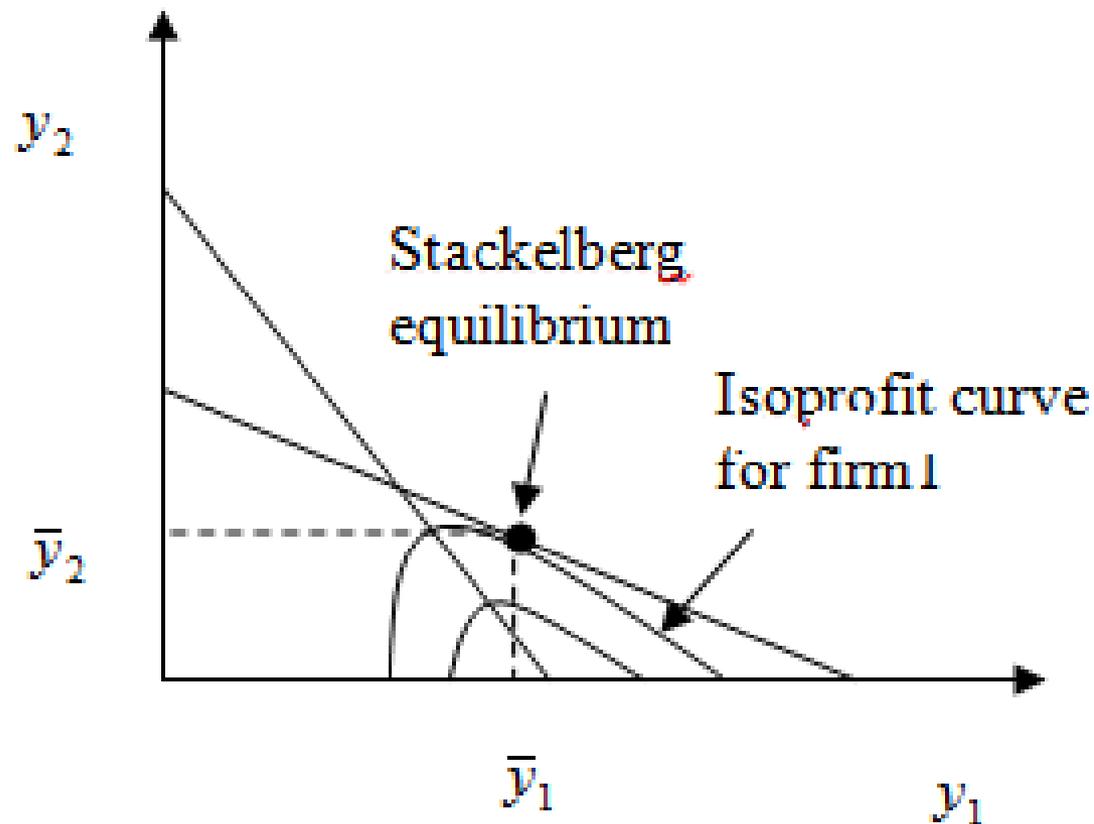
$$\max_{y_1} \pi_1(y_1, y_2) = p(y_1 + y_2)y_1 - c_1(y_1)$$

such that

$$y_2 = f_2(y_1)$$

$$\max_{y_1} p(y_1 + f_2(y_1))y_1 - c_1(y_1)$$

- The Stackelberg equilibrium – Firm 1, the leader, chooses the point on the firm 2's reaction curve that touches firm 1's lowest possible isoprofit line, thus yielding the highest possible profits for firm 1.



## Example 1

Suppose that we have two firms that face linear demand curve

$p = 100 - \frac{1}{2}(y_1 + y_2)$ , and their cost functions are

$c_1(y_1) = 5y_1$  and  $c_2(y_2) = \frac{1}{2}y_2^2$  respectively.

- Calculate the Cournot equilibrium amount of output for each firm and firms' profits.
- If firm 2 behaves as a follower and firm 1 behaves as a leader, calculate the Stackelberg equilibrium amount of output for each firm and firms' profits.
- If firm 1 behaves as a follower and firm 2 behaves as a leader, calculate the Stackelberg equilibrium amount of output for each firm and firms' profits.

The Cournot equilibrium

$$\pi_1(y_1, y_2) = [100 - \frac{1}{2}(y_1 + y_2)]y_1 - 5y_1 = 95y_1 - \frac{1}{2}y_1^2 - \frac{1}{2}y_1y_2$$

$$\pi_2(y_1, y_2) = [100 - \frac{1}{2}(y_1 + y_2)]y_2 - \frac{1}{2}y_2^2 = 100y_2 - \frac{1}{2}y_1y_2 - \frac{1}{2}y_2^2$$

The first-order conditions are the following

$$\begin{cases} \frac{\partial \pi_1(y_1, y_2)}{\partial y_1} = 0 \\ \frac{\partial \pi_2(y_1, y_2)}{\partial y_2} = 0 \end{cases} \Rightarrow \begin{cases} y_1 + \frac{1}{2}y_2 = 95 \\ \frac{1}{2}y_1 + 2y_2 = 100 \end{cases}$$

The solution

$$\bar{y}_1 = 80, \quad \bar{y}_2 = 30, \quad p = 45, \quad \pi_1 = 3200, \quad \pi_2 = 900.$$

## The Stackelberg equilibrium

1. Firm 2 behaves as a follower

$$\pi_2(y_1, y_2) = [100 - \frac{1}{2}(y_1 + y_2)]y_2 - \frac{1}{2}y_2^2 = 100y_2 - \frac{1}{2}y_1y_2 - \frac{1}{2}y_2^2$$

$$\frac{\partial \pi_2(y_1, y_2)}{\partial y_2} = 0 \quad \Rightarrow \quad \frac{1}{2}y_1 + 2y_2 = 100 \quad \Rightarrow \quad y_2 = 50 - \frac{1}{4}y_1$$

The leader's problem

$$\max_{y_1} \pi_1(y_1, y_2) = 95y_1 - \frac{1}{2}y_1^2 - \frac{1}{2}y_1y_2$$

such that

$$y_2 = 50 - \frac{1}{4}y_1$$

$$\pi_1(y_1) = 95y_1 - \frac{1}{2}y_1^2 - \frac{1}{2}y_1\left(50 - \frac{1}{4}y_1\right) = 70y_1 - \frac{3}{8}y_1^2$$

$$\frac{d\pi_1(y_1)}{dy_1} = 70 - \frac{3}{4}y_1 = 0 \quad \Rightarrow \quad \bar{y}_1 = \frac{280}{3} = 93.3, \quad \bar{y}_2 = \frac{80}{3} = 26.7$$

$$p = 40, \quad \pi_1 = 3266.7, \quad \pi_2 = 711.1$$

2. Firm 1 behaves as a follower

$$\pi_1(y_1, y_2) = [100 - \frac{1}{2}(y_1 + y_2)]y_1 - 5y_1 = 95y_1 - \frac{1}{2}y_1^2 - \frac{1}{2}y_1y_2$$

$$\frac{\partial \pi_1(y_1, y_2)}{\partial y_1} = 0 \quad \Rightarrow \quad y_1 + \frac{1}{2}y_2 = 95 \quad \Rightarrow \quad y_1 = 95 - \frac{1}{2}y_2$$

The leader's problem

$$\max_{y_2} \pi_2(y_1, y_2) = 100y_2 - \frac{1}{2}y_1y_2 - y_2^2$$

such that

$$y_1 = 95 - \frac{1}{2}y_2$$

$$\pi_2(y_2) = 100y_2 - \frac{1}{2}(95 - \frac{1}{2}y_2)y_2 - y_2^2 = \frac{105}{2}y_2 - \frac{3}{4}y_2^2$$

$$\frac{d\pi_2(y_2)}{dy_2} = \frac{105}{2} - \frac{3}{2}y_2 = 0 \quad \Rightarrow \quad \bar{y}_2 = 35, \quad \bar{y}_1 = 77.5$$

$$p = 43.75, \quad \pi_1 = 3003.13, \quad \pi_2 = 918.75$$

- **Partial equilibrium:** how demand and supply are affected by the price of the particular good.
- **General equilibrium:** how demand and supply conditions interact in several markets to determine the prices of many goods.
- General equilibrium refers to the study of how the economy can adjust to have demand equal supply in all markets at the same time.

# Market Equilibrium

There are 2 traders and 2 commodities.

Each trader has the initial endowment of commodities,

$a_i^j$  - the amount of commodity  $i$  with trader  $j$ :

$$a^1 = (a_1^1, a_2^1) \quad , \quad a^2 = (a_1^2, a_2^2)$$

Traders have utilities on commodity bundles:

$u^1(x_1, x_2)$  and  $u^2(x_1, x_2)$ , respectively.

Traders come to market and exchange commodities to maximize their utilities. Commodities are divisible.

$x_i^j$  - the amount of commodity  $i$  with trader  $j$  after the trade:

$$x^1 = (x_1^1, x_2^1) \quad , \quad x^2 = (x_1^2, x_2^2) \quad . \quad \text{Commodity } i \text{ has a price } p_i \quad .$$

No deficiency or surplus of any commodity

$$\forall i \quad \sum_{j=1}^2 x_i^j = \sum_{j=1}^2 a_i^j ;$$

$$\forall j \quad \sum_{i=1}^2 p_i \cdot x_i^j = \sum_{i=1}^2 p_i \cdot a_i^j = I^j$$

$$\max_{x_1, x_2} u^1(x_1, x_2)$$

such that

$$p_1 x_1 + p_2 x_2 = I^1 = p_1 a_1^1 + p_2 a_2^1$$

$$x^1(p) = (x_1^1(p), x_2^1(p))$$

$$\max_{x_1, x_2} u^2(x_1, x_2)$$

such that

$$p_1 x_1 + p_2 x_2 = I^2 = p_1 a_1^2 + p_2 a_2^2$$

$$x^2(p) = (x_1^2(p), x_2^2(p))$$

The equilibrium can be described as a set of prices  $(\bar{p}_1, \bar{p}_2)$  such that

$$x_1^1(\bar{p}_1, \bar{p}_2) + x_1^2(\bar{p}_1, \bar{p}_2) = a_1^1 + a_1^2 \quad \bar{p} = \lambda(\bar{p}_1, \bar{p}_2)$$

$$x_2^1(\bar{p}_1, \bar{p}_2) + x_2^2(\bar{p}_1, \bar{p}_2) = a_2^1 + a_2^2$$

$$z(\bar{p}) = 0$$

aggregate excess demand function  $z(\bar{p}) = x(\bar{p}) - a$

## Example 2

Let utilities of traders have the form  $u^1(x_1, x_2) = x_1^2 x_2$  and  $u^2(x_1, x_2) = x_1^{1/3} x_2^{1/2}$ . Their initial endowments are the following  $a^1 = (10, 10)$  and  $a^2 = (20, 5)$ .

$$\max_{x_1, x_2} x_1^2 x_2$$

subject to

$$p_1 x_1 + p_2 x_2 = I^1 = 10p_1 + 10p_2$$

$$x^1(p) = \left( \frac{20p_1 + 20p_2}{3p_1}, \frac{10p_1 + 10p_2}{3p_2} \right)$$

$$\max_{x_1, x_2} x_1^{1/3} x_2^{1/2}$$

subject to

$$p_1 x_1 + p_2 x_2 = I^2 = 20p_1 + 5p_2$$

$$x^2(p) = \left( \frac{8p_1 + 2p_2}{p_1}, \frac{12p_1 + 3p_2}{p_2} \right)$$

$$x(p) = x^1(p) + x^2(p) = \left( \frac{44p_1 + 26p_2}{3p_1}, \frac{46p_1 + 19p_2}{3p_2} \right)$$

$$a = a^1 + a^2 = (30, 15)$$

$$z(p) = x(p) - a = \left( \frac{-46p_1 + 26p_2}{3p_1}, \frac{46p_1 - 26p_2}{3p_2} \right)$$

$$z(\bar{p}) = 0 \quad \Rightarrow \quad \bar{p} = \lambda(13, 23)$$

$$x^1(\bar{p}) = \left( \frac{20 \cdot 13 + 20 \cdot 23}{3 \cdot 13}, \frac{120}{23} \right) = (18.46, 5.22)$$

$$u^1(10,10) = 10^2 \cdot 10 = 1000$$

$$u^1\left(\frac{240}{13}, \frac{120}{23}\right) = 1788.24$$

$$x^2(\bar{p}) = \left( \frac{150}{13}, \frac{225}{23} \right) = (11.54, 9.78)$$

$$u^2(20,5) = 6.07$$

$$u^2\left(\frac{150}{13}, \frac{225}{23}\right) = 7.07$$

## Example 3

Let utilities of traders have the form  $u^1(x_1, x_2) = x_1^{\frac{1}{2}} + x_2^{\frac{1}{2}}$

and  $u^2(x_1, x_2) = \min\left\{\frac{x_1}{2}, x_2\right\}$ .

Their initial endowments are the following

$a^1 = (10, 10)$  and  $a^2 = (20, 5)$ .

$$\max_{x_1, x_2} \left( x_1^{\frac{1}{2}} + x_2^{\frac{1}{2}} \right)$$

subject to

$$p_1 x_1 + p_2 x_2 = I^1 = 10p_1 + 10p_2$$

$$x^1(p) = \left( \frac{10p_2}{p_1}, \frac{10p_1}{p_2} \right)$$

$$\min \left\{ \frac{x_1}{2}, x_2 \right\}$$

$$p_1 x_1 + p_2 x_2 = I^2 = 20p_1 + 5p_2$$

$$x^2(p) = \left( \frac{40p_1 + 10p_2}{2p_1 + p_2}, \frac{20p_1 + 5p_2}{2p_1 + p_2} \right)$$

$$x(p) = x^1(p) + x^2(p) = \left( \frac{40p_1^2 + 30p_1p_2 + 10p_2^2}{p_1(2p_1 + p_2)}, \frac{20p_1^2 + 30p_1p_2 + 5p_2^2}{p_2(2p_1 + p_2)} \right)$$

$$a = a^1 + a^2 = (30, 15)$$

$$z(p) = x(p) - a = \left( \frac{-20p_1^2 + 10p_2^2}{p_1(2p_1 + p_2)}, \frac{20p_1^2 - 10p_2^2}{p_2(2p_1 + p_2)} \right)$$

$$z(\bar{p}) = 0 \quad \Rightarrow \quad \bar{p} = \lambda(1, \sqrt{2})$$

$$x^1(\bar{p}) = \left( \frac{10\sqrt{2}}{1}, \frac{10 \cdot 1}{\sqrt{2}} \right) = (14.14, 7.07)$$

$$u^1(10,10) = 2\sqrt{10} = 6.32$$

$$u^1(14.14, 7.07) = 6.42$$

$$x^2(\bar{p}) = \left( \frac{40 + 10\sqrt{2}}{2 + \sqrt{2}}, \frac{20 + 5\sqrt{2}}{2 + \sqrt{2}} \right) = (15.86, 7.93)$$

$$u^2(20,5) = \min\left\{ \frac{20}{2}, 5 \right\} = 5$$

$$u^2(15.86, 7.93) = 7.93$$

## Example 4

Let utilities of traders be the following:

$$u^1(x_1, x_2, x_3) = \min\{x_1, x_3\}, \quad u^2(x_1, x_2, x_3) = \min\{x_1, x_2\},$$

$$u^3(x_1, x_2, x_3) = \min\{x_2, x_3\}.$$

The traders have the initial endowments

$$a^1 = (1, 0, 0), \quad a^2 = (0, 1, 0), \quad a^3 = (0, 0, 1), \text{ respectively.}$$